Constraint Programming

Lecture 5:
2-way Branching
Global Arc Consistency
Last Lecture: Constraint Graphs

- Each **node** represents a variable.
- Constraints are represented by **edges**.
- Also known as the **primal** graph representation.

Variables \{x_1, x_2, x_3\}
- Domain of each is \{0,1\}
- Constraints: \{x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3\}

What did we say if the constraints are non-binary?
Last Lecture: Dual Representation

- Transform each non-binary constraint into a **dual variable**.
  - Has an associated domain of the allowed tuples.
- Add **binary constraints** between any two dual variables that share an original variable.
  - Constraint insists that the assignments to the two nodes that it connects agree for the shared original variables.
Last Lecture: Consistency

- Consistency Properties:
  - Node Consistency.
  - Arc Consistency.
  - Local/Global.
- We split each undirected edge representing a constraint into **two directional arcs**:

  \[
  \overset{x_2}{\rightarrow} \overset{x_3}{\leftarrow} \text{ arc}(x_2, x_3) \\
  \overset{x_2}{\leftarrow} \overset{x_3}{\rightarrow} \text{ arc}(x_3, x_2)
  \]

- Concept of **support**.
- Arc revision.
Last Lecture: Forward Checking

• After an assignment to $x_i$, revise arcs from every future variable to $x_i$ once.
  • I.e. Enforce local arc consistency on a subset of the arcs.
This Lecture

• D-way vs. 2-way branching revisited.
• Enforcing global arc consistency.
D-way vs. 2-way
Branching Revisited
d-way vs 2-way Branching

- In d-way branching, each branch under a parent node represents the assignment of one of d domain values from the domain of a particular variable.
- E.g. if $x$ in $\{1, 2, 3\}$:

$$x = 1 \quad x = 2 \quad x = 3$$

- Try extending partial assignment with $x = v$ first. If no solution remove $v$ from consideration, continue:

$$\text{Root: original CSP}$$

$$x = v \quad x \neq v$$

$$\text{sub-CSP} \quad \text{sub-CSP}$$
d-way vs 2-way: Spotting the Difference

- Solve the following instance using Forward Checking, variable ordering $x_1$, $x_2$, $x_3$, and both d-way and 2-way branching:

\[
x_1 \in \{1..50\}
\]

\[
x_2 \in \{1\}
\]

\[
x_3 \in \{1\}
\]
2-way Branching

- After $x_1$ is assigned 1, the dead end is spotted as soon as we try and assign $x_2$.
- After 1 is removed from $D_1$, FC immediately spots that there is no support for the sole value in $D_2$ or $D_3$. 

Lecture 5: Global Arc Consistency
d-way Branching

- After $x_1$ is assigned 1, the dead end is spotted as soon as we try and assign $x_2$.
- When we assign $x_1$ any other value, we spot the dead end immediately (49 more times!)

![Diagram of d-way Branching]

$x_1 = 1$, $x_1 = 2$, $x_1 = 3$, $x_1 = 50$

Lecture 5: Global Arc Consistency
d-way vs 2-way

• We could have had to do a lot more work at each of those 49 dead ends.
  • Especially if we had a more powerful algorithm than FC.
• So from now on: 2-way branching.
Forward Checking: 2-way Version

• The main recursive procedure:

Procedure ForwardChecking(varList):
    if (completeAssignment())
        printSolution()
        exit()
    var = select(varList)
    val = select(domain(var))
    branchFCLeft(varList, var, val)
    branchFCRight(varList, var, val)
Forward Checking: 2-way Version

- How to branch left:

**Procedure** `branchFCLeft(varList, depth, val)`:

assign($x_{\text{depth}}$, $val$)

If `reviseFutureArcs(depth)`:

ForwardChecking(`varList-x_{\text{depth}}`)  
undoPruning()

unassign($x_{\text{depth}}$, $val$)

- The `undoPruning` procedure reverses the changes made by `reviseFutureArcs`. 
Forward Checking: 2-way Version

• How to branch right:

Procedure branchFCRight(varList, depth, val):
  deleteValue(x_{depth}, val)
  If domain(x_{depth}) not empty:
    If reviseFutureArcs(depth):
      ForwardChecking(varList-x_{depth})
  undoPruning()
  restoreValue(x_{depth}, val)

• The undoPruning procedure reverses the changes made by reviseFutureArcs.
Forward Checking: 2-way Version

• How to prune future domains:

Procedure reviseFutureArcs(depth):
  consistent = true
  For future = depth+1 To n While consistent
    consistent = revise(arc(x_{future}, x_{depth}))
    // Prunes domain \( D_{future} \)
  return consistent

• consistent becomes false iff a domain is emptied by revise
Enforcing Global Arc Consistency

AC1
Dechter p57
Enforcing Global Arc Consistency

• Recall that global arc consistency means that every arc in the problem is arc consistent.
• We know that can’t enforce it in one pass in general (see last lecture).
• Need an algorithm that efficiently re-revises arcs as necessary.
AC1 (This Isn’t It)

• The simplest method of achieving global arc consistency.
• Iterate passes of the revision of every arc until there are no more changes.
• This is obviously inefficient:
  • All arcs are revised, regardless of the possibility of them having become inconsistent.
AC1: Example

$x_1, D_1 = \{2, 10, 16\}$

$x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$

$x_4, D_4 = \{2, 5, 10, 11\}$

• Arcs:
  - $\text{arc}(x_1, x_2)$, $\text{arc}(x_1, x_3)$, $\text{arc}(x_1, x_4)$, $\text{arc}(x_2, x_1)$,
  - $\text{arc}(x_2, x_3)$, $\text{arc}(x_2, x_4)$, $\text{arc}(x_3, x_1)$, $\text{arc}(x_3, x_2)$,
  - $\text{arc}(x_3, x_4)$, $\text{arc}(x_4, x_1)$, $\text{arc}(x_4, x_2)$, $\text{arc}(x_4, x_3)$
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10, 16\} \quad < \quad x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\} \quad > \quad x_4, D_4 = \{2, 5, 10, 11\} \]

• Arcs:
  
  - \( \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \)
  
  - \( \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \)
  
  - \( \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)

No Changes to \( D_1 \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10, 16\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- **Arcs:**
  - \( \text{arc}(x_1, x_2) \), \( \text{arc}(x_1, x_3) \), \( \text{arc}(x_1, x_4) \), \( \text{arc}(x_2, x_1) \), \( \text{arc}(x_2, x_3) \), \( \text{arc}(x_2, x_4) \), \( \text{arc}(x_3, x_1) \), \( \text{arc}(x_3, x_2) \), \( \text{arc}(x_3, x_4) \), \( \text{arc}(x_4, x_1) \), \( \text{arc}(x_4, x_2) \), \( \text{arc}(x_4, x_3) \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \quad \text{No Changes to } D_1 \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- **Arcs:**
  - \( \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

• Arcs:
  - \( \text{arc}(x_1, x_2) \), \( \text{arc}(x_1, x_3) \), \( \text{arc}(x_1, x_4) \), \( \text{arc}(x_2, x_1) \),
  - \( \text{arc}(x_2, x_3) \), \( \text{arc}(x_2, x_4) \), \( \text{arc}(x_3, x_1) \), \( \text{arc}(x_3, x_2) \),
  - \( \text{arc}(x_3, x_4) \), \( \text{arc}(x_4, x_1) \), \( \text{arc}(x_4, x_2) \), \( \text{arc}(x_4, x_3) \)

No Changes to \( D_2 \)
AC1 Example: Iteration 1

$x_1, D_1 = \{2, 10\}$  
$x_2, D_2 = \{9, 12, 21\}$  
$x_3, D_3 = \{9, 10, 11\}$  
$x_4, D_4 = \{2, 5, 10, 11\}$

• Arcs:
  - arc($x_1, x_2$), arc($x_1, x_3$), arc($x_1, x_4$), arc($x_2, x_1$),
    arc($x_2, x_3$), arc($x_2, x_4$), arc($x_3, x_1$), arc($x_3, x_2$),
    arc($x_3, x_4$), arc($x_4, x_1$), arc($x_4, x_2$), arc($x_4, x_3$)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- Arcs:
  - arc(\(x_1, x_2\)), arc(\(x_1, x_3\)), arc(\(x_1, x_4\)), arc(\(x_2, x_1\)),
  - arc(\(x_2, x_3\)), \textcolor{red}{\textbf{arc}(x_2, x_4)}), arc(\(x_3, x_1\)), arc(\(x_3, x_2\)),
  - arc(\(x_3, x_4\)), arc(\(x_4, x_1\)), arc(\(x_4, x_2\)), arc(\(x_4, x_3\))

\textbf{No Changes to} \(D_2\)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

No Changes to \( D_3 \)

• Arcs:
  - \( \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- Arcs:
  - \( \text{arc}(x_1, x_2) \), \( \text{arc}(x_1, x_3) \), \( \text{arc}(x_1, x_4) \), \( \text{arc}(x_2, x_1) \),
  - \( \text{arc}(x_2, x_3) \), \( \text{arc}(x_2, x_4) \), \( \text{arc}(x_3, x_1) \), \( \text{arc}(x_3, x_2) \),
  - \( \text{arc}(x_3, x_4) \), \( \text{arc}(x_4, x_1) \), \( \text{arc}(x_4, x_2) \), \( \text{arc}(x_4, x_3) \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

**No Changes to** \( D_3 \)

**Arcs:**

- \( \text{arc}(x_1, x_2) \), \( \text{arc}(x_1, x_3) \), \( \text{arc}(x_1, x_4) \), \( \text{arc}(x_2, x_1) \), \( \text{arc}(x_2, x_3) \), \( \text{arc}(x_2, x_4) \), \( \text{arc}(x_3, x_1) \), \( \text{arc}(x_3, x_2) \), \( \text{arc}(x_3, x_4) \), \( \text{arc}(x_4, x_1) \), \( \text{arc}(x_4, x_2) \), \( \text{arc}(x_4, x_3) \)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- **Arcs:**
  - \(\text{arc}(x_1, x_2)\), \(\text{arc}(x_1, x_3)\), \(\text{arc}(x_1, x_4)\), \(\text{arc}(x_2, x_1)\),
  - \(\text{arc}(x_2, x_3)\), \(\text{arc}(x_2, x_4)\), \(\text{arc}(x_3, x_1)\), \(\text{arc}(x_3, x_2)\),
  - \(\text{arc}(x_3, x_4)\), \(\text{arc}(x_4, x_1)\), \(\text{arc}(x_4, x_2)\), \(\text{arc}(x_4, x_3)\)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{5, 10, 11\} \]

- **Arcs:**
  - arc\((x_1, x_2)\), arc\((x_1, x_3)\), arc\((x_1, x_4)\), arc\((x_2, x_1)\),
  - arc\((x_2, x_3)\), arc\((x_2, x_4)\), arc\((x_3, x_1)\), arc\((x_3, x_2)\),
  - arc\((x_3, x_4)\), arc\((x_4, x_1)\), **arc**\((x_4, x_2)\), arc\((x_4, x_3)\)
AC1 Example: Iteration 1

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10, 11\} \]

- Arcs:
  - \( \text{arc}(x_1, x_2) \)
  - \( \text{arc}(x_1, x_3) \)
  - \( \text{arc}(x_1, x_4) \)
  - \( \text{arc}(x_2, x_1) \)
  - \( \text{arc}(x_2, x_3) \)
  - \( \text{arc}(x_2, x_4) \)
  - \( \text{arc}(x_3, x_1) \)
  - \( \text{arc}(x_3, x_2) \)
  - \( \text{arc}(x_3, x_4) \)
  - \( \text{arc}(x_4, x_1) \)
  - \( \text{arc}(x_4, x_2) \)
  - \( \text{arc}(x_4, x_3) \)

Iteration 1 Complete
AC1 Example: Iteration 2

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

- **Arcs:**
  - \[ \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \]
  - \[ \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \]
  - \[ \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \]
...AC1 Example: Iteration 2

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

- Arcs:
  - \( \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)

Iteration 2 Complete – but still need Iteration 3!
AC1: Summary

• AC1 never really existed. It is an artificially stupid method of enforcing global arc consistency.

• It was named as such in the following classic paper:

• However, Mackworth’s paper also contained a much better algorithm…
AC3

Dechter p58
What Happened to AC2?

• AC2 (aka Waltz’s Algorithm) is equivalent to AC3 (which we’re about to see), but its presentation is less clear.

• Make a local subset of the arcs consistent.
• Add the arcs related to a new variable to the subset.
• Perform all revisions necessary to maintain consistency.
• Do until no more variables to add.
• Avoids redundant work.
Procedure AC3()

For $i = 1$ To $n$ NC($x_i$)

$Q =$ all arcs($x_i, x_j$)

While Not empty($Q$)

Remove any arc($x_i, x_j$) in $Q$

If Revise(arc($x_i, x_j$))

Add to $Q$ all arcs($x_h, x_i$) ($h \neq j$)
AC3

Procedure AC3()

- For $i = 1$ To $n$ NC($x_i$)

  $Q =$ all arcs($x_i, x_j$)

While Not empty($Q$)

  Remove any arc($x_i, x_j$) in $Q$

  If Revise(arc($x_i, x_j$))

    Add to $Q$ all arcs($x_h, x_i$) ($h \neq j$)

- Global node consistency is a precondition.
AC3

Procedure AC3()

For $i = 1$ To $n$ NC($x_i$)

$Q =$ all arcs($x_i$, $x_j$)

While Not empty($Q$)

Remove any arc($x_i$, $x_j$) in $Q$

If Revise(arc($x_i$, $x_j$))

Add to $Q$ all arcs($x_h$, $x_i$) ($h \neq j$)

• Begin by initialising a queue with all arcs in the instance.
AC3

Procedure AC3()

For \( i = 1 \) To \( n \) NC\((x_i)\)

\( Q = \) all arcs\((x_i, x_j)\)

While Not empty\((Q)\)

Remove any arc\((x_i, x_j)\) in \( Q \)

If Revise\((\text{arc}(x_i, x_j))\)

Add to \( Q \) all arcs\((x_h, x_i)\) \((h \neq j)\)

• Termination condition: no arcs left to revise. At this point global arc consistency holds.
AC3

Procedure AC3()

For $i = 1$ To $n$ NC($x_i$)

$Q =$ all arcs($x_i, x_j$)

While Not empty($Q$)

- Remove any arc($x_i, x_j$) in $Q$

If Revise(arc($x_i, x_j$))

Add to $Q$ all arcs($x_h, x_i$) ($h \neq j$)

- Order of processing is unimportant to final result.
- But can result in more/less work.
AC3

Procedure AC3()

For $i = 1$ To $n$ NC($x_i$)

$Q = \text{all arcs}(x_i, x_j)$

While Not empty($Q$)

Remove any arc($x_i, x_j$) in $Q$

If Revise(arc($x_i, x_j$))

Add to $Q$ all arcs($x_h, x_i$) ($h \neq j$)

• Assuming that Revise returns true if $D_i$ has been changed.
• Assuming also that empty domains are caught.
AC3

Procedure AC3()

For $i = 1$ To $n$ NC($x_i$)

$Q =$ all arcs($x_i, x_j$)

While Not empty($Q$)

Remove any arc($x_i, x_j$) in $Q$

If Revise(arc($x_i, x_j$))

Add to $Q$ all arcs($x_h, x_i$) ($h \neq j$)

• If $D_i$ changes, must re-revise arcs incident on $x_i$.
• Assumption: if arc already present, not added.
AC3

Procedure AC3()

For \( i = 1 \) To \( n \) NC(\( x_i \))

\( Q = \) all arcs(\( x_i, x_j \))

While Not empty(\( Q \))

Remove any arc(\( x_i, x_j \)) in \( Q \)

If Revise(arc(\( x_i, x_j \)))

Add to \( Q \) all arcs(\( x_h, x_i \)) \((h \neq j)\)

• Exception: Don’t add arc(\( x_j, x_i \)).
When to Add to the Queue

• Revising arc($x_1, x_2$).

$x_1, D_1 = \{2, 11, 16\}$ \(\xrightarrow{<} \) \(\bullet \) \(\quad\) \(\bullet \quad x_2, D_2 = \{5, 10, 11\}\)

• Gives:

$x_1, D_1 = \{2\}$ \(\xrightarrow{<} \) \(\bullet \quad\) \(\bullet \quad x_2, D_2 = \{5, 10, 11\}\)

• Why don’t we need to add to arc($x_2, x_1$) if it’s not already in the queue?
When to Add to the Queue

- Revising arc($x_1, x_2$)

\[ x_1, D_1 = \{2, 11, 16\} \xrightarrow{<} x_2, D_2 = \{5, 10, 11\} \]

- Gives:

\[ x_1, D_1 = \{2\} \xrightarrow{<} x_2, D_2 = \{5, 10, 11\} \]

- 11, 16 were removed because they had no support in $D_2$.

- Support is bi-directional.

- So if they were not supported, they were not supporting.
AC3: Example

- Initial Queue:
  - $\text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_1, x_4), \text{arc}(x_2, x_1),$
  - $\text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2),$
  - $\text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3)$

$x_1, D_1 = \{2, 10, 16\}$

$x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$

$x_4, D_4 = \{2, 5, 10, 11\}$
AC3: Example

$x_1, D_1 = \{2, 10, 16\}$ \hspace{2cm} $x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$ \hspace{2cm} $x_4, D_4 = \{2, 5, 10, 11\}$

- Queue:
  
  - $\text{arc}(x_1, x_2)$, $\text{arc}(x_1, x_3)$, $\text{arc}(x_1, x_4)$, $\text{arc}(x_2, x_1)$, $\text{arc}(x_2, x_3)$, $\text{arc}(x_2, x_4)$, $\text{arc}(x_3, x_1)$, $\text{arc}(x_3, x_2)$, $\text{arc}(x_3, x_4)$, $\text{arc}(x_4, x_1)$, $\text{arc}(x_4, x_2)$, $\text{arc}(x_4, x_3)$

  - No Changes to $D_1$: Nothing to add to Q
AC3: Example

\[ x_1, D_1 = \{2, 10, 16\}, \quad x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\}, \quad x_4, D_4 = \{2, 5, 10, 11\} \]

- **Queue:**
  - arc\((x_1, x_3)\), arc\((x_1, x_4)\), arc\((x_2, x_1)\), arc\((x_2, x_3)\),
    arc\((x_2, x_4)\), arc\((x_3, x_1)\), arc\((x_3, x_2)\), arc\((x_3, x_4)\),
    arc\((x_4, x_1)\), arc\((x_4, x_2)\), arc\((x_4, x_3)\)
  
  Add to Q: arc\((x_2, x_1)\), arc\((x_4, x_1)\) — both already there
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- **Queue:**
  - \( \text{arc}(x_1, x_4), \text{arc}(x_2, x_1), \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)

No Changes to \( D_1 \): Nothing to add to \( Q \)
AC3: Example

\( x_1, D_1 = \{2, 10\} \quad > \quad x_2, D_2 = \{9, 12, 21\} \)

\( x_3, D_3 = \{9, 10, 11\} \quad > \quad x_4, D_4 = \{2, 5, 10, 11\} \)

- Queue:
  - No Changes to \( D_2 \): Nothing to add to Q
  - \( \text{arc}(x_2, x_1), \text{arc}(x_2, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3) \)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

Add to Q: arc(\(x_1, x_2\)), arc(\(x_4, x_2\)) – latter already there

\[ \text{Queue:} \]

- \(\text{arc}(x_2, x_3)\), \(\text{arc}(x_2, x_4)\), \(\text{arc}(x_3, x_1)\), \(\text{arc}(x_3, x_2)\), \(\text{arc}(x_3, x_4)\), \(\text{arc}(x_4, x_1)\), \(\text{arc}(x_4, x_2)\), \(\text{arc}(x_4, x_3)\), \(\text{arc}(x_1, x_2)\)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \quad < \quad x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{9, 10, 11\} \quad > \quad x_4, D_4 = \{2, 5, 10, 11\} \]

- Queue:
  - No Changes to \( D_2 \): Nothing to add to \( Q \)
  - \( \text{arc}(x_2, x_4), \text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3), \text{arc}(x_1, x_2) \)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- Queue:
  
  - No Changes to \(D_3\): Nothing to add to \(Q\)
  
  - \(\text{arc}(x_3, x_1), \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3), \text{arc}(x_1, x_2)\)
AC3: Example

\( x_1, D_1 = \{2, 10\} \)
\( x_2, D_2 = \{9\} \)
\( x_3, D_3 = \{9, 10, 11\} \)
\( x_4, D_4 = \{2, 5, 10, 11\} \)

**Queue:**
- \( \text{arc}(x_1, x_3), \text{arc}(x_4, x_3) \) – latter already there
- \( \text{arc}(x_3, x_2), \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3), \text{arc}(x_1, x_2), \text{arc}(x_1, x_3) \)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \quad x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \quad x_4, D_4 = \{2, 5, 10, 11\} \]

- Queue: No Changes to \( D_3 \): Nothing to add to Q

- \( \text{arc}(x_3, x_4), \text{arc}(x_4, x_1), \text{arc}(x_4, x_2), \text{arc}(x_4, x_3), \text{arc}(x_1, x_2), \text{arc}(x_1, x_3) \)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

- Queue: Add to Q: arc\((x_2, x_4), \text{arc}(x_3, x_4)\)
  - arc\((x_4, x_1)\), arc\((x_4, x_2)\), arc\((x_4, x_3)\), arc\((x_1, x_2)\),
    arc\((x_1, x_3)\), arc\((x_2, x_4)\), arc\((x_3, x_4)\)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{5, 10, 11\} \]

\[ \text{Add to } Q: \text{arc}(x_1, x_4), \text{arc}(x_3, x_4) \]
\[ \text{latter already present} \]

- Queue:
  - \( \text{arc}(x_4, x_2) \), \( \text{arc}(x_4, x_3) \), \( \text{arc}(x_1, x_2) \), \( \text{arc}(x_1, x_3) \),
  \( \text{arc}(x_2, x_4) \), \( \text{arc}(x_3, x_4) \), \( \text{arc}(x_1, x_4) \)
AC3: Example

- Queue:
  - \( \text{arc}(x_1, x_4), \text{arc}(x_2, x_4) \) – both already present
  - \( \text{arc}(x_4, x_3), \text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_4), \text{arc}(x_1, x_4) \)

- \( x_1, D_1 = \{2, 10\} \)
- \( x_2, D_2 = \{9\} \)
- \( x_3, D_3 = \{10, 11\} \)
- \( x_4, D_4 = \{10, 11\} \)
AC3: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

Queue:

- Add to Q: arc\(x_3, x_1\), arc\(x_4, x_1\)
- \(\text{arc}(x_1, x_2), \text{arc}(x_1, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_4), \text{arc}(x_1, x_4), \text{arc}(x_3, x_1), \text{arc}(x_4, x_1)\)
AC3: Example

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

- Queue:
  - No Changes to \( D_1 \): Nothing to add to Q
  - \( \text{arc}(x_1, x_3), \text{arc}(x_2, x_4), \text{arc}(x_3, x_4), \text{arc}(x_1, x_4), \text{arc}(x_3, x_1), \text{arc}(x_4, x_1) \)
AC3: Example

- Queue:
  - $\text{arc}(x_2, x_4)$, $\text{arc}(x_3, x_4)$, $\text{arc}(x_1, x_4)$, $\text{arc}(x_3, x_1)$, $\text{arc}(x_4, x_1)$

No Changes to $D_2$: Nothing to add to Q

$x_1, D_1 = \{2\}$

$x_2, D_2 = \{9\}$

$x_3, D_3 = \{10, 11\}$

$x_4, D_4 = \{10\}$
AC3: Example

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

- Queue:
  - Add to Q: \( \text{arc}(x_1, x_3), \text{arc}(x_2, x_3) \)
  - \( \text{arc}(x_3, x_4), \text{arc}(x_1, x_4), \text{arc}(x_3, x_1), \text{arc}(x_4, x_1), \text{arc}(x_1, x_3), \text{arc}(x_2, x_3) \)
AC3: Example

\( x_1, D_1 = \{2\} \)

\( x_2, D_2 = \{9\} \)

\( x_3, D_3 = \{11\} \)

\( x_4, D_4 = \{10\} \)

• Queue:
  - No Changes to \( D_1 \): Nothing to add to Q
  - \( \text{arc}(x_1, x_4), \text{arc}(x_3, x_1), \text{arc}(x_4, x_1), \text{arc}(x_1, x_3), \text{arc}(x_2, x_3) \)
AC3: Example

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{11\} \]
\[ x_4, D_4 = \{10\} \]

- **Queue:** No Changes to \( D_3 \): Nothing to add to Q
- \( \text{arc}(x_3, x_1), \text{arc}(x_4, x_1), \text{arc}(x_1, x_3), \text{arc}(x_2, x_3) \)
AC3: Example

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{11\} \]
\[ x_4, D_4 = \{10\} \]

• Queue:
  - No Changes to \( D_4 \): Nothing to add to Q
  - \( \text{arc}(x_4, x_1) \), \( \text{arc}(x_1, x_3) \), \( \text{arc}(x_2, x_3) \)
AC3: Example

- Queue: No Changes to $D_1$: Nothing to add to Q
- $\text{arc}(x_1, x_3), \text{arc}(x_2, x_3)$
AC3: Example

\[ x_1, D_1 = \{2\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{11\} \]

\[ x_4, D_4 = \{10\} \]

- Queue: No Changes to \( D_2 \): Nothing to add to Q

- \( \text{arc}(x_2, x_3) \)
AC3: Example

- $x_1, D_1 = \{2\}$
- $x_2, D_2 = \{9\}$
- $x_3, D_3 = \{11\}$
- $x_4, D_4 = \{10\}$

- **Queue:**
  - **Empty**
- **CSP is arc consistent.**
AC1 vs. AC3

• On this problem AC1 took 36 revisions.
• AC3 took 21:
  • NB the constraint graph is fully connected, so this problem is a worst case for AC3 in terms of connectivity.
Complexity Considerations
AC3: Time Complexity

- AC3 has worst-case time complexity of $O(ed^3)$.
  - $e$ is the number of edges in the constraint graph.
  - $d$ is the maximum domain size.
- Each arc $(x_i, x_j)$ is revised iff it enters the queue.
  - This only happens if $D_j$ loses a value.
  - Each arc can enter the queue/be revised at most $d$ times.
  - There are $2e$ arcs, so Revise() is executed $O(ed)$ times.
  - Complexity of Revise() is $O(d^2)$. 
AC3: Time Complexity

• Improving this bound has been the focus of much research effort.
  • How can we do better?
Coarse-grained vs. Fine-grained

• AC3 is smart about which arcs might need to be revised (coarse-grained)
• But it doesn’t know how the support of individual domain elements has been affected by a revision.
  • If we have this fine-grained knowledge, can save even more work.
AC4

Dechter p60
AC4 – A fine-grained Algorithm

• A **counter** is associated with each arc-label pair 
\( \langle \text{arc}(x_i, x_j), d_i \rangle \), where \( d_i \) in \( D_i \).
  • Records the number of elements of \( D_j \) that support \( d_i \).

• A **set** of pairs \( \langle x_i, d_i \rangle \) is associated with each \( d_j \) in \( D_j \).
  • Records the assignments for which \( d_j \) in \( D_j \) provides support.

• For each domain element, we know how much support it has, and for which other elements it provides support.
AC4 – A fine-grained Algorithm

- Maintain:
  - A table of deleted domain elements: $M[x_i, d_i]$ is 1 if deleted, 0 otherwise.
  - A list, $L$, of deletions to propagate.

- $L$ is similar to $Q$ in AC3
  - It contains a list of things to be processed
  - However, $L$ contains individual value deletions rather than arcs
AC4 – A fine-grained Algorithm

• Algorithm has 2 phases:
  – Initialise
  – Propagate

• Presentation is very slightly different to Dechter; these slides are your reference.
AC4: Initialise

• Process each arc once.
• Enumerate support for each element of $D_i$.
• If no support for $x_i = d_i$, then it is pruned.
• Initialise list of deletions to propagate.
AC4 Initialise: Example

$x_1, D_1 = \{2, 10, 16\}$

$x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$

$x_4, D_4 = \{2, 5, 10, 11\}$

Counter
AC4 Initialise: Example

\( x_1, D_1 = \{2, 10, 16\} \)
\( x_2, D_2 = \{9, 12, 21\} \)
\( x_3, D_3 = \{9, 10, 11\} \)
\( x_4, D_4 = \{2, 5, 10, 11\} \)

Nothing supported yet

| \( x_1, 2 \) | \{\} | \( x_3, 10 \) | \{\} |
| \( x_1, 10 \) | \{\} | \( x_3, 11 \) | \{\} |
| \( x_1, 16 \) | \{\} | \( x_4, 2 \) | \{\} |
| \( x_2, 9 \) | \{\} | \( x_4, 5 \) | \{\} |
| \( x_2, 12 \) | \{\} | \( x_4, 10 \) | \{\} |
| \( x_2, 21 \) | \{\} | \( x_4, 11 \) | \{\} |
| \( x_3, 9 \) | \{\} |

Nothing Deleted yet

| \( x_1, 2 \) | 0 | \( x_3, 10 \) | 0 |
| \( x_1, 10 \) | 0 | \( x_3, 11 \) | 0 |
| \( x_1, 16 \) | 0 | \( x_4, 2 \) | 0 |
| \( x_2, 9 \) | 0 | \( x_4, 5 \) | 0 |
| \( x_2, 12 \) | 0 | \( x_4, 10 \) | 0 |
| \( x_2, 21 \) | 0 | \( x_4, 11 \) | 0 |
| \( x_3, 9 \) | 0 |
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10, 16\} \quad x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\} \quad x_4, D_4 = \{2, 5, 10, 11\} \]

Counter

<table>
<thead>
<tr>
<th>(\text{arc}(x_1, x_2)), 2</th>
<th>3</th>
<th>(\text{arc}(x_1, x_4)), 16</th>
<th>0</th>
<th>(\text{arc}(x_2, x_4)), 12</th>
<th>0</th>
<th>(\text{arc}(x_3, x_4)), 9</th>
<th>0</th>
<th>(\text{arc}(x_4, x_2)), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{arc}(x_1, x_2)), 10</td>
<td>0</td>
<td>(\text{arc}(x_2, x_1)), 9</td>
<td>0</td>
<td>(\text{arc}(x_2, x_4)), 21</td>
<td>0</td>
<td>(\text{arc}(x_3, x_4)), 10</td>
<td>0</td>
<td>(\text{arc}(x_4, x_2)), 10</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_2)), 16</td>
<td>0</td>
<td>(\text{arc}(x_2, x_1)), 12</td>
<td>0</td>
<td>(\text{arc}(x_3, x_1)), 9</td>
<td>0</td>
<td>(\text{arc}(x_3, x_4)), 11</td>
<td>0</td>
<td>(\text{arc}(x_4, x_2)), 11</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 2</td>
<td>0</td>
<td>(\text{arc}(x_2, x_1)), 21</td>
<td>0</td>
<td>(\text{arc}(x_3, x_1)), 10</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 2</td>
<td>0</td>
<td>(\text{arc}(x_4, x_3)), 2</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 10</td>
<td>0</td>
<td>(\text{arc}(x_2, x_3)), 9</td>
<td>0</td>
<td>(\text{arc}(x_3, x_1)), 11</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 5</td>
<td>0</td>
<td>(\text{arc}(x_4, x_3)), 5</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 16</td>
<td>0</td>
<td>(\text{arc}(x_2, x_3)), 12</td>
<td>0</td>
<td>(\text{arc}(x_3, x_2)), 9</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 10</td>
<td>0</td>
<td>(\text{arc}(x_4, x_3)), 10</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_4)), 2</td>
<td>0</td>
<td>(\text{arc}(x_2, x_3)), 21</td>
<td>0</td>
<td>(\text{arc}(x_3, x_2)), 10</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 11</td>
<td>0</td>
<td>(\text{arc}(x_4, x_3)), 11</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_4)), 10</td>
<td>0</td>
<td>(\text{arc}(x_2, x_4)), 9</td>
<td>0</td>
<td>(\text{arc}(x_3, x_2)), 11</td>
<td>0</td>
<td>(\text{arc}(x_4, x_2)), 2</td>
<td>0</td>
<td>(\text{arc}(x_4, x_2)), 2</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ S[x_2, 9] = \{\langle x_1, 2 \rangle\}, \ S[x_2, 12] = \{\langle x_1, 2 \rangle\}, \ S[x_2, 21] = \{\langle x_1, 2 \rangle\} \]
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10, 16\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

\[ S[x_2, 12] = \{\langle x_1, 2\rangle, \langle x_1, 10\rangle\}, \quad S[x_2, 21] = \{\langle x_1, 2\rangle, \langle x_1, 10\rangle\} \]
### AC4 Initialise: Example

$x_1, D_1 = \{2, 10, 16\}$

$x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$

$x_4, D_4 = \{2, 5, 10, 11\}$

#### Counter

<table>
<thead>
<tr>
<th></th>
<th>$\text{arc}(x_1, x_2)$</th>
<th>2</th>
<th>$\text{arc}(x_1, x_4)$</th>
<th>16</th>
<th>$\text{arc}(x_2, x_4)$</th>
<th>12</th>
<th>$\text{arc}(x_3, x_4)$</th>
<th>9</th>
<th>$\text{arc}(x_4, x_2)$</th>
<th>5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{arc}(x_1, x_2)$</td>
<td>10</td>
<td>2</td>
<td>$\text{arc}(x_2, x_1)$</td>
<td>9</td>
<td>$\text{arc}(x_2, x_4)$</td>
<td>21</td>
<td>$\text{arc}(x_3, x_4)$</td>
<td>10</td>
<td>$\text{arc}(x_4, x_2)$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_2)$</td>
<td>16</td>
<td>1</td>
<td>$\text{arc}(x_2, x_1)$</td>
<td>12</td>
<td>$\text{arc}(x_3, x_1)$</td>
<td>9</td>
<td>$\text{arc}(x_3, x_4)$</td>
<td>11</td>
<td>$\text{arc}(x_4, x_2)$</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_3)$</td>
<td>2</td>
<td>0</td>
<td>$\text{arc}(x_2, x_1)$</td>
<td>21</td>
<td>$\text{arc}(x_3, x_1)$</td>
<td>10</td>
<td>$\text{arc}(x_4, x_1)$</td>
<td>2</td>
<td>$\text{arc}(x_4, x_3)$</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_3)$</td>
<td>10</td>
<td>0</td>
<td>$\text{arc}(x_2, x_3)$</td>
<td>9</td>
<td>$\text{arc}(x_3, x_1)$</td>
<td>11</td>
<td>$\text{arc}(x_4, x_1)$</td>
<td>5</td>
<td>$\text{arc}(x_4, x_3)$</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_3)$</td>
<td>16</td>
<td>0</td>
<td>$\text{arc}(x_2, x_3)$</td>
<td>12</td>
<td>$\text{arc}(x_3, x_2)$</td>
<td>9</td>
<td>$\text{arc}(x_4, x_1)$</td>
<td>10</td>
<td>$\text{arc}(x_4, x_3)$</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_4)$</td>
<td>2</td>
<td>0</td>
<td>$\text{arc}(x_2, x_3)$</td>
<td>21</td>
<td>$\text{arc}(x_3, x_2)$</td>
<td>10</td>
<td>$\text{arc}(x_4, x_1)$</td>
<td>11</td>
<td>$\text{arc}(x_4, x_3)$</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>$\text{arc}(x_1, x_4)$</td>
<td>10</td>
<td>0</td>
<td>$\text{arc}(x_2, x_4)$</td>
<td>9</td>
<td>$\text{arc}(x_3, x_2)$</td>
<td>11</td>
<td>$\text{arc}(x_4, x_2)$</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$S[x_2, 21] = \{\langle x_1, 2 \rangle, \langle x_1, 10 \rangle, \langle x_1, 16 \rangle\}$
AC4 Initialise: Example

\(x_1, D_1 = \{2, 10, 16\}\) \(x_2, D_2 = \{9, 12, 21\}\)

\(x_3, D_3 = \{9, 10, 11\}\) \(x_4, D_4 = \{2, 5, 10, 11\}\)

Counter

<table>
<thead>
<tr>
<th></th>
<th>arc((x_1, x_2)), 2</th>
<th>arc((x_1, x_4)), 16</th>
<th>arc((x_2, x_4)), 12</th>
<th>arc((x_3, x_4)), 9</th>
<th>arc((x_4, x_2)), 5</th>
<th>arc((x_4, x_3)), 2</th>
<th>arc((x_4, x_1)), 5</th>
<th>arc((x_3, x_1)), 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc((x_1, x_2)), 10</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_2, x_1)), 12</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 16</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_2, x_1)), 21</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 16</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_1, x_4)), 2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>arc((x_2, x_4)), 9</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\(S[x_3, 9] = \{\langle x_1, 2 \rangle\}\), \(S[x_3, 10] = \{\langle x_1, 2 \rangle\}\), \(S[x_3, 11] = \{\langle x_1, 2 \rangle\}\)
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10, 16\} \]

\[ x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\} \]

\[ x_4, D_4 = \{2, 5, 10, 11\} \]

Counter

\[
\begin{array}{cccccccc}
\text{arc}(x_1, x_2), & 2 & \text{arc}(x_1, x_4), & 16 & \text{arc}(x_2, x_4), & 12 & \text{arc}(x_3, x_4), & 9 \\
\text{arc}(x_1, x_2), & 10 & \text{arc}(x_2, x_1), & 9 & \text{arc}(x_2, x_4), & 21 & \text{arc}(x_3, x_4), & 10 \\
\text{arc}(x_1, x_2), & 16 & \text{arc}(x_2, x_1), & 12 & \text{arc}(x_3, x_1), & 9 & \text{arc}(x_3, x_4), & 11 \\
\text{arc}(x_1, x_3), & 2 & \text{arc}(x_2, x_1), & 21 & \text{arc}(x_3, x_1), & 10 & \text{arc}(x_4, x_1), & 2 \\
\text{arc}(x_1, x_3), & 10 & \text{arc}(x_2, x_3), & 9 & \text{arc}(x_3, x_1), & 11 & \text{arc}(x_4, x_1), & 5 \\
\text{arc}(x_1, x_3), & 16 & \text{arc}(x_2, x_3), & 12 & \text{arc}(x_3, x_2), & 9 & \text{arc}(x_4, x_1), & 10 \\
\text{arc}(x_1, x_4), & 2 & \text{arc}(x_2, x_3), & 21 & \text{arc}(x_3, x_2), & 10 & \text{arc}(x_4, x_1), & 11 \\
\text{arc}(x_1, x_4), & 10 & \text{arc}(x_2, x_4), & 9 & \text{arc}(x_3, x_2), & 11 & \text{arc}(x_4, x_2), & 2 \\
\end{array}
\]

\[ S[x_3, 11] = \{\langle x_1, 2 \rangle, \langle x_1, 10 \rangle\} \]
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10, 16\} \]

\[ x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\} \]

\[ x_4, D_4 = \{2, 5, 10, 11\} \]

\[ M[x_1, 16] = 1 \]
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9, 12, 21\} \]

\[ x_3, D_3 = \{9, 10, 11\} \]

\[ x_4, D_4 = \{2, 5, 10, 11\} \]

Counter

<table>
<thead>
<tr>
<th>arc(x_1, x_2), 2</th>
<th>3</th>
<th>arc(x_1, x_4), 16</th>
<th>0</th>
<th>arc(x_2, x_4), 12</th>
<th>0</th>
<th>arc(x_3, x_4), 9</th>
<th>0</th>
<th>arc(x_4, x_2), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc(x_1, x_2), 10</td>
<td>2</td>
<td>arc(x_2, x_1), 9</td>
<td>0</td>
<td>arc(x_2, x_4), 21</td>
<td>0</td>
<td>arc(x_3, x_4), 10</td>
<td>0</td>
<td>arc(x_4, x_2), 10</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_2), 16</td>
<td>1</td>
<td>arc(x_2, x_1), 12</td>
<td>0</td>
<td>arc(x_3, x_1), 9</td>
<td>0</td>
<td>arc(x_3, x_4), 11</td>
<td>0</td>
<td>arc(x_4, x_2), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_3), 2</td>
<td>3</td>
<td>arc(x_2, x_1), 21</td>
<td>0</td>
<td>arc(x_3, x_1), 10</td>
<td>0</td>
<td>arc(x_4, x_1), 2</td>
<td>0</td>
<td>arc(x_4, x_3), 2</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_3), 10</td>
<td>1</td>
<td>arc(x_2, x_3), 9</td>
<td>0</td>
<td>arc(x_3, x_1), 11</td>
<td>0</td>
<td>arc(x_4, x_1), 5</td>
<td>0</td>
<td>arc(x_4, x_3), 5</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_3), 16</td>
<td>0</td>
<td>arc(x_2, x_3), 12</td>
<td>0</td>
<td>arc(x_3, x_2), 9</td>
<td>0</td>
<td>arc(x_4, x_1), 10</td>
<td>0</td>
<td>arc(x_4, x_3), 10</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_4), 2</td>
<td>3</td>
<td>arc(x_2, x_3), 21</td>
<td>0</td>
<td>arc(x_3, x_2), 10</td>
<td>0</td>
<td>arc(x_4, x_1), 11</td>
<td>0</td>
<td>arc(x_4, x_3), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc(x_1, x_4), 10</td>
<td>0</td>
<td>arc(x_2, x_4), 9</td>
<td>0</td>
<td>arc(x_3, x_2), 11</td>
<td>0</td>
<td>arc(x_4, x_2), 2</td>
<td>0</td>
<td>arc(x_4, x_2), 2</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ S[x_4, 5] = \{\langle x_1, 2\rangle\}, S[x_4, 10] = \{\langle x_1, 2\rangle\}, S[x_4, 11] = \{\langle x_1, 2\rangle\} \]
AC4 Initialise: Example

$x_1, D_1 = \{2, 10\}$

$x_2, D_2 = \{9, 12, 21\}$

$x_3, D_3 = \{9, 10, 11\}$

$x_4, D_4 = \{2, 5, 10, 11\}$

Counter

<table>
<thead>
<tr>
<th>arc($x_1, x_2$), 2</th>
<th>3</th>
<th>arc($x_1, x_4$), 16</th>
<th>0</th>
<th>arc($x_2, x_4$), 12</th>
<th>0</th>
<th>arc($x_3, x_4$), 9</th>
<th>0</th>
<th>arc($x_4, x_2$), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc($x_1, x_2$), 10</td>
<td>2</td>
<td>arc($x_2, x_1$), 9</td>
<td>0</td>
<td>arc($x_2, x_4$), 21</td>
<td>0</td>
<td>arc($x_3, x_4$), 10</td>
<td>0</td>
<td>arc($x_4, x_2$), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_2$), 16</td>
<td>1</td>
<td>arc($x_2, x_1$), 12</td>
<td>0</td>
<td>arc($x_3, x_1$), 9</td>
<td>0</td>
<td>arc($x_3, x_4$), 11</td>
<td>0</td>
<td>arc($x_4, x_2$), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 2</td>
<td>3</td>
<td>arc($x_2, x_1$), 21</td>
<td>0</td>
<td>arc($x_3, x_1$), 10</td>
<td>0</td>
<td>arc($x_4, x_1$), 2</td>
<td>0</td>
<td>arc($x_4, x_3$), 2</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 10</td>
<td>1</td>
<td>arc($x_2, x_3$), 9</td>
<td>0</td>
<td>arc($x_3, x_1$), 11</td>
<td>0</td>
<td>arc($x_4, x_1$), 5</td>
<td>0</td>
<td>arc($x_4, x_3$), 5</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 16</td>
<td>0</td>
<td>arc($x_2, x_3$), 12</td>
<td>0</td>
<td>arc($x_3, x_2$), 9</td>
<td>0</td>
<td>arc($x_4, x_1$), 10</td>
<td>0</td>
<td>arc($x_4, x_3$), 10</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_4$), 2</td>
<td>3</td>
<td>arc($x_2, x_3$), 21</td>
<td>0</td>
<td>arc($x_3, x_2$), 10</td>
<td>0</td>
<td>arc($x_4, x_1$), 11</td>
<td>0</td>
<td>arc($x_4, x_3$), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_4$), 10</td>
<td>1</td>
<td>arc($x_2, x_4$), 9</td>
<td>0</td>
<td>arc($x_3, x_2$), 11</td>
<td>0</td>
<td>arc($x_4, x_2$), 2</td>
<td>0</td>
<td>arc($x_4, x_3$), 11</td>
<td>0</td>
</tr>
</tbody>
</table>

$S[x_4, 11] = \{\langle x_1, 2\rangle, \langle x_1, 10\rangle\}$
AC4 Initialise: Example

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9, 12, 21\} \]
\[ x_3, D_3 = \{9, 10, 11\} \]
\[ x_4, D_4 = \{2, 5, 10, 11\} \]

**Counter**

<table>
<thead>
<tr>
<th>arc(x, y)</th>
<th>3</th>
<th>arc(x, y), 12</th>
<th>0</th>
<th>arc(x, y), 9</th>
<th>0</th>
<th>arc(x, y), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc(x, y)</td>
<td>2</td>
<td>arc(x, y), 9</td>
<td>0</td>
<td>arc(x, y), 21</td>
<td>0</td>
<td>arc(x, y), 10</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>1</td>
<td>arc(x, y), 12</td>
<td>0</td>
<td>arc(x, y), 1, 9</td>
<td>0</td>
<td>arc(x, y), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>2</td>
<td>arc(x, y), 21</td>
<td>0</td>
<td>arc(x, y), 10</td>
<td>0</td>
<td>arc(x, y), 2</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>1</td>
<td>arc(x, y), 9</td>
<td>0</td>
<td>arc(x, y), 11</td>
<td>0</td>
<td>arc(x, y), 5</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>0</td>
<td>arc(x, y), 12</td>
<td>0</td>
<td>arc(x, y), 9</td>
<td>0</td>
<td>arc(x, y), 10</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>3</td>
<td>arc(x, y), 21</td>
<td>0</td>
<td>arc(x, y), 10</td>
<td>0</td>
<td>arc(x, y), 11</td>
<td>0</td>
</tr>
<tr>
<td>arc(x, y)</td>
<td>1</td>
<td>arc(x, y), 9</td>
<td>0</td>
<td>arc(x, y), 11</td>
<td>0</td>
<td>arc(x, y), 2</td>
<td>0</td>
</tr>
</tbody>
</table>

- 16 has been deleted, ignore this arc/value pair.
AC4 Initialisation Complete

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

**Counter**

<table>
<thead>
<tr>
<th>( \text{arc}(x_1, x_2) )</th>
<th>2</th>
<th>3</th>
<th>( \text{arc}(x_2, x_1) )</th>
<th>9</th>
<th>1</th>
<th>( \text{arc}(x_3, x_1) )</th>
<th>10</th>
<th>2</th>
<th>( \text{arc}(x_4, x_2) )</th>
<th>5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{arc}(x_1, x_2) ), 10</td>
<td>2</td>
<td>( \text{arc}(x_2, x_1) ), 9</td>
<td>1</td>
<td>( \text{arc}(x_3, x_1) ), 10</td>
<td>2</td>
<td>( \text{arc}(x_4, x_2) ), 5</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_2) ), 16</td>
<td>1</td>
<td>( \text{arc}(x_2, x_1) ), 12</td>
<td>2</td>
<td>( \text{arc}(x_3, x_1) ), 9</td>
<td>1</td>
<td>( \text{arc}(x_4, x_2) ), 10</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 2</td>
<td>3</td>
<td>( \text{arc}(x_2, x_1) ), 21</td>
<td>2</td>
<td>( \text{arc}(x_3, x_1) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_4, x_1) ), 2</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_2, x_3) ), 9</td>
<td>2</td>
<td>( \text{arc}(x_3, x_1) ), 11</td>
<td>2</td>
<td>( \text{arc}(x_4, x_1) ), 5</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 16</td>
<td>0</td>
<td>( \text{arc}(x_2, x_3) ), 12</td>
<td>0</td>
<td>( \text{arc}(x_3, x_2) ), 9</td>
<td>0</td>
<td>( \text{arc}(x_4, x_1) ), 10</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_4) ), 2</td>
<td>3</td>
<td>( \text{arc}(x_2, x_3) ), 21</td>
<td>0</td>
<td>( \text{arc}(x_3, x_2) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_4, x_1) ), 11</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_4) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_2, x_4) ), 9</td>
<td>2</td>
<td>( \text{arc}(x_3, x_2) ), 11</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ L = \{ \langle x_1, 16 \rangle, \langle x_2, 12 \rangle, \langle x_2, 21 \rangle, \langle x_3, 9 \rangle, \langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle \} \]
AC4: Example

\( x_1, D_1 = \{2, 10, 16\} \)
\( x_2, D_2 = \{9, 12, 21\} \)
\( x_3, D_3 = \{9, 10, 11\} \)
\( x_4, D_4 = \{2, 5, 10, 11\} \)

- Initialise already pruned these elements.
AC4: Propagate

• Choose a deleted value to propagate from \( L \) (order unimportant to final result).
• Iterate over the assignments supported by the deleted value.
• Decrement their counters.
• If counter reaches 0 and element not already deleted (How could be already deleted?), delete and add to \( L \).
• Repeat until Global AC established. (\( L \) empty).
AC4: Propagate

- Choose a deleted value to propagate from $L$ (order unimportant to final result).
- Iterate over the assignments supported by the deleted value.
- Decrement their counters.
- If counter reaches 0 and element not already deleted (lost support on another arc), delete and add to $L$.
- Repeat until Global AC established. ($L$ empty).
**AC4: Propagate**

$x_1, D_1 = \{2, 10\}$

$x_2, D_2 = \{9\}$

$x_3, D_3 = \{10, 11\}$

$x_4, D_4 = \{10\}$

Counter

$L = \{\langle x_1, 16 \rangle, \langle x_2, 12 \rangle, \langle x_2, 21 \rangle, \langle x_3, 9 \rangle, \langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle\}$
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

### Counter

<table>
<thead>
<tr>
<th>Counter</th>
<th>arc((x_1, x_2)), 2</th>
<th>arc((x_1, x_2)), 10</th>
<th>arc((x_1, x_3)), 2</th>
<th>arc((x_1, x_3)), 10</th>
<th>arc((x_1, x_4)), 2</th>
<th>arc((x_1, x_4)), 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>arc((x_2, x_1)), 9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>arc((x_2, x_1)), 12</td>
<td>arc((x_3, x_1)), 9</td>
<td></td>
<td>arc((x_3, x_4)), 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>arc((x_3, x_1)), 10</td>
<td>arc((x_4, x_1)), 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>arc((x_3, x_1)), 11</td>
<td>arc((x_4, x_1)), 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>arc((x_3, x_2)), 9</td>
<td>arc((x_3, x_2)), 10</td>
<td>arc((x_4, x_1)), 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>arc((x_2, x_3)), 21</td>
<td></td>
<td></td>
<td></td>
<td>arc((x_2, x_4)), 9</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>arc((x_2, x_3)), 12</td>
<td>arc((x_3, x_2)), 9</td>
<td>arc((x_3, x_2)), 10</td>
<td>arc((x_4, x_3)), 10</td>
<td>arc((x_4, x_3)), 11</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>arc((x_3, x_2)), 10</td>
<td>arc((x_4, x_1)), 11</td>
<td>arc((x_4, x_3)), 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>arc((x_3, x_2)), 10</td>
<td>arc((x_4, x_1)), 11</td>
<td>arc((x_4, x_3)), 11</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(S\left[ x_1, 16 \right] = \{ \} \) – 16 wasn’t supporting anything (deleted early).
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th>(\text{arc}(x_1, x_2))</th>
<th>2 #2</th>
<th>(\text{arc}(x_2, x_1))</th>
<th>9 #2</th>
<th>(\text{arc}(x_1, x_3))</th>
<th>2 #2</th>
<th>(\text{arc}(x_2, x_2))</th>
<th>10 #2</th>
<th>(\text{arc}(x_1, x_4))</th>
<th>2 #2</th>
<th>(\text{arc}(x_4, x_2))</th>
<th>5 #0</th>
<th>(\text{arc}(x_3, x_1))</th>
<th>3 #0</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td></td>
<td>2</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td></td>
<td>3</td>
<td></td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

\[ L = \{\langle x_2, 12 \rangle, \langle x_2, 21 \rangle, \langle x_3, 9 \rangle, \langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle\} \]
AC4: Propagate

\(x_1, D_1 = \{2, 10\}\)

\(x_2, D_2 = \{9\}\)

\(x_3, D_3 = \{10, 11\}\)

\(x_4, D_4 = \{10\}\)

\[S[x_2, 12] = \{\langle x_1, 2 \rangle, \langle x_1, 10 \rangle\}\] – Decrement counters.
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

Counter

\[ S[x_2, 12] = \{\langle x_1, 2\rangle, \langle x_1, 10\rangle\} \rightarrow \text{None zero: no deletions.} \]

Lecture 5: Global Arc Consistency
AC4: Propagate

$x_1, D_1 = \{2, 10\}$

$x_3, D_3 = \{10, 11\}$

$x_2, D_2 = \{9\}$

$x_4, D_4 = \{10\}$

Counter

<table>
<thead>
<tr>
<th>arc($x_1, x_2$), 2</th>
<th>2</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>arc($x_4, x_2$), 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc($x_1, x_2$), 10</td>
<td>1</td>
<td>arc($x_2, x_1$), 9</td>
<td>1</td>
<td>arc($x_3, x_4$), 10</td>
<td>2</td>
</tr>
<tr>
<td>arc($x_1, x_2$), 16</td>
<td>1</td>
<td>arc($x_2, x_1$), 12</td>
<td>2</td>
<td>arc($x_3, x_1$), 9</td>
<td>1</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 2</td>
<td>3</td>
<td>arc($x_2, x_1$), 21</td>
<td>2</td>
<td>arc($x_3, x_1$), 10</td>
<td>1</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 10</td>
<td>1</td>
<td>arc($x_2, x_3$), 9</td>
<td>2</td>
<td>arc($x_3, x_1$), 11</td>
<td>2</td>
</tr>
<tr>
<td>arc($x_1, x_3$), 16</td>
<td>0</td>
<td>arc($x_2, x_3$), 12</td>
<td>0</td>
<td>arc($x_3, x_2$), 9</td>
<td>0</td>
</tr>
<tr>
<td>arc($x_1, x_4$), 2</td>
<td>3</td>
<td>arc($x_2, x_3$), 21</td>
<td>0</td>
<td>arc($x_3, x_2$), 10</td>
<td>1</td>
</tr>
<tr>
<td>arc($x_1, x_4$), 10</td>
<td>1</td>
<td>arc($x_2, x_4$), 9</td>
<td>2</td>
<td>arc($x_3, x_2$), 11</td>
<td>1</td>
</tr>
</tbody>
</table>

$L = \{\langle x_2, 21 \rangle, \langle x_3, 9 \rangle, \langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle\}$

Lecture 5: Global Arc Consistency
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th>( \text{arc}(x_1, x_2) )</th>
<th>( \text{arc}(x_1, x_3) )</th>
<th>( \text{arc}(x_1, x_4) )</th>
<th>( \text{arc}(x_2, x_1) )</th>
<th>( \text{arc}(x_2, x_3) )</th>
<th>( \text{arc}(x_2, x_4) )</th>
<th>( \text{arc}(x_3, x_1) )</th>
<th>( \text{arc}(x_3, x_2) )</th>
<th>( \text{arc}(x_3, x_4) )</th>
<th>( \text{arc}(x_4, x_1) )</th>
<th>( \text{arc}(x_4, x_2) )</th>
<th>( \text{arc}(x_4, x_3) )</th>
<th>( \text{arc}(x_4, x_4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>10</td>
<td>16</td>
<td>9</td>
<td>2</td>
<td>3</td>
<td>10</td>
<td>12</td>
<td>21</td>
<td>10</td>
<td>2</td>
<td>12</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>21</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

\( S[x_2, 21] = \{\langle x_1, 2\rangle, \langle x_1, 10\rangle, \langle x_1, 16\rangle\} \) – Decrement counters
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \] \[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \] \[ x_4, D_4 = \{10\} \]

<table>
<thead>
<tr>
<th>\text{Counter}</th>
<th>\text{arc}(x_1, x_2), 2</th>
<th>1</th>
<th>\text{arc}(x_2, x_1), 9</th>
<th>1</th>
<th>\text{arc}(x_3, x_4), 10</th>
<th>2</th>
<th>\text{arc}(x_4, x_2), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc(x_1, x_2), 10</td>
<td>1</td>
<td>arc(x_2, x_1), 9</td>
<td>1</td>
<td>arc(x_3, x_4), 10</td>
<td>2</td>
<td>arc(x_4, x_2), 5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>arc(x_1, x_2), 16</td>
<td>1</td>
<td>arc(x_2, x_1), 12</td>
<td>2</td>
<td>arc(x_3, x_1), 9</td>
<td>1</td>
<td>arc(x_3, x_4), 11</td>
<td>3</td>
<td>arc(x_4, x_2), 11</td>
</tr>
<tr>
<td>arc(x_1, x_3), 2</td>
<td>3</td>
<td>arc(x_2, x_1), 21</td>
<td>2</td>
<td>arc(x_3, x_1), 10</td>
<td>1</td>
<td>arc(x_4, x_1), 2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>arc(x_1, x_3), 10</td>
<td>1</td>
<td>arc(x_2, x_3), 9</td>
<td>2</td>
<td>arc(x_3, x_1), 11</td>
<td>2</td>
<td>arc(x_4, x_1), 5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>arc(x_1, x_3), 16</td>
<td>0</td>
<td>arc(x_2, x_3), 12</td>
<td>0</td>
<td>arc(x_3, x_2), 9</td>
<td>0</td>
<td>arc(x_4, x_1), 10</td>
<td>1</td>
<td>arc(x_4, x_3), 10</td>
</tr>
<tr>
<td>arc(x_1, x_4), 2</td>
<td>3</td>
<td>arc(x_2, x_3), 21</td>
<td>0</td>
<td>arc(x_3, x_2), 10</td>
<td>1</td>
<td>arc(x_4, x_1), 11</td>
<td>2</td>
<td>arc(x_4, x_3), 11</td>
</tr>
<tr>
<td>arc(x_1, x_4), 10</td>
<td>1</td>
<td>arc(x_2, x_4), 9</td>
<td>2</td>
<td>arc(x_3, x_2), 11</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2 is still supported by 9.
AC4: Propagate

\[ x_1, D_1 = \{2, 10\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th>Arc</th>
<th>Counter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc((x_1, x_2), 2)</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_2), 10)</td>
<td>0</td>
<td>arc((x_2, x_1), 9)</td>
</tr>
<tr>
<td>arc((x_1, x_2), 16)</td>
<td>1</td>
<td>arc((x_2, x_1), 21)</td>
</tr>
<tr>
<td>arc((x_1, x_3), 2)</td>
<td>3</td>
<td>arc((x_3, x_1), 10)</td>
</tr>
<tr>
<td>arc((x_1, x_3), 10)</td>
<td>1</td>
<td>arc((x_3, x_1), 11)</td>
</tr>
<tr>
<td>arc((x_1, x_3), 16)</td>
<td>0</td>
<td>arc((x_3, x_2), 10)</td>
</tr>
<tr>
<td>arc((x_1, x_4), 2)</td>
<td>3</td>
<td>arc((x_3, x_2), 21)</td>
</tr>
<tr>
<td>arc((x_1, x_4), 10)</td>
<td>1</td>
<td>arc((x_3, x_2), 11)</td>
</tr>
</tbody>
</table>

10 no longer has support: delete it, append \(\langle x_1, 10 \rangle\) to \(L\).
AC4: Propagate

\[ x_1, D_1 = \{2\} \]

\[ \Rightarrow \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ \Rightarrow \]

\[ x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th>arc((x_1, x_2)), 2</th>
<th>1</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>(\text{arc}(x_4, x_2), 5)</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc((x_1, x_2)), 10</td>
<td>0</td>
<td>arc((x_2, x_1)), 9</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td>(\text{arc}(x_3, x_4), 10)</td>
<td>2</td>
</tr>
<tr>
<td>arc((x_1, x_2)), 16</td>
<td>0</td>
<td>arc((x_2, x_1)), 12</td>
<td>2</td>
<td>arc((x_3, x_1)), 9</td>
<td>1</td>
<td>arc((x_3, x_4)), 11</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_3)), 2</td>
<td>3</td>
<td>arc((x_2, x_1)), 21</td>
<td>2</td>
<td>arc((x_3, x_1)), 10</td>
<td>1</td>
<td>arc((x_4, x_1)), 2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_3)), 10</td>
<td>1</td>
<td>arc((x_2, x_3)), 9</td>
<td>2</td>
<td>arc((x_3, x_1)), 11</td>
<td>2</td>
<td>arc((x_4, x_1)), 5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_3)), 16</td>
<td>0</td>
<td>arc((x_2, x_3)), 12</td>
<td>0</td>
<td>arc((x_3, x_2)), 9</td>
<td>0</td>
<td>arc((x_4, x_1)), 10</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_4)), 2</td>
<td>3</td>
<td>arc((x_2, x_3)), 21</td>
<td>0</td>
<td>arc((x_3, x_2)), 10</td>
<td>1</td>
<td>arc((x_4, x_1)), 11</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_4)), 10</td>
<td>1</td>
<td>arc((x_2, x_4)), 9</td>
<td>2</td>
<td>arc((x_3, x_2)), 11</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

16 no longer has support on this arc – but it has already been deleted.
AC4: Propagate

\[ x_1, D_1 = \{2\} \quad \text{<} \quad x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \quad \text{<} \quad x_4, D_4 = \{10\} \]

Counter

\[
\begin{array}{cccccc}
\text{arc}(x_1, x_2), 2 & 1 & & & & \text{arc}(x_4, x_2), 5 & 0 \\
\text{arc}(x_1, x_2), 10 & 0 & \text{arc}(x_2, x_1), 9 & 1 & & \text{arc}(x_3, x_4), 10 & 2 \\
\text{arc}(x_1, x_2), 16 & 0 & \text{arc}(x_2, x_1), 12 & 2 & \text{arc}(x_3, x_1), 9 & 1 \\
\text{arc}(x_1, x_3), 2 & 3 & \text{arc}(x_2, x_1), 21 & 2 & \text{arc}(x_3, x_1), 10 & 1 \\
\text{arc}(x_1, x_3), 10 & 1 & \text{arc}(x_2, x_3), 9 & 2 & \text{arc}(x_3, x_1), 11 & 2 \\
\text{arc}(x_1, x_3), 16 & 0 & \text{arc}(x_2, x_3), 12 & 0 & \text{arc}(x_3, x_2), 9 & 0 \\
\text{arc}(x_1, x_4), 2 & 3 & \text{arc}(x_2, x_3), 21 & 0 & \text{arc}(x_3, x_2), 10 & 1 \\
\text{arc}(x_1, x_4), 10 & 1 & \text{arc}(x_2, x_4), 9 & 2 & \text{arc}(x_3, x_2), 11 & 1 \\
\end{array}
\]

\[ L = \{\langle x_3, 9 \rangle, \langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle, \langle x_1, 10 \rangle\} \]
**AC4: Propagate**

\[ x_1, D_1 = \{2\} \]
\[ x_2, D_2 = \{9\} \]
\[ x_3, D_3 = \{10, 11\} \]
\[ x_4, D_4 = \{10\} \]

### Counter

<table>
<thead>
<tr>
<th>Arc</th>
<th>Value</th>
<th>Counter Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{arc}(x_1, x_2) ), 2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_2) ), 10</td>
<td>0</td>
<td>( \text{arc}(x_2, x_1) ), 9</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_2) ), 16</td>
<td>0</td>
<td>( \text{arc}(x_2, x_1) ), 12</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 2</td>
<td>3</td>
<td>( \text{arc}(x_2, x_1) ), 21</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_2, x_3) ), 9</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_3) ), 16</td>
<td>0</td>
<td>( \text{arc}(x_2, x_3) ), 12</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_4) ), 2</td>
<td>3</td>
<td>( \text{arc}(x_2, x_3) ), 21</td>
</tr>
<tr>
<td>( \text{arc}(x_1, x_4) ), 10</td>
<td>1</td>
<td>( \text{arc}(x_2, x_4) ), 9</td>
</tr>
</tbody>
</table>

\[ \mathcal{S}[x_3, 9] = \{\langle x_1, 2 \rangle\} \] – Decrement counters
AC4: Propagate

\[ x_1, D_1 = \{2\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

Counter

\[
\begin{array}{cccc|cc}
\text{arc}(x_1, x_2), 2 & 1 & & & \text{arc}(x_1, x_2), 10 & 0 \\
\text{arc}(x_1, x_2), 10 & 0 & \text{arc}(x_2, x_1), 9 & 1 & \text{arc}(x_3, x_4), 10 & 2 \\
\text{arc}(x_1, x_2), 16 & 0 & \text{arc}(x_2, x_1), 12 & 2 & \text{arc}(x_3, x_1), 9 & 1 \\
\text{arc}(x_1, x_3), 2 & 2 & \text{arc}(x_2, x_1), 21 & 2 & \text{arc}(x_3, x_1), 10 & 1 \\
\text{arc}(x_1, x_3), 10 & 1 & \text{arc}(x_2, x_3), 9 & 2 & \text{arc}(x_3, x_1), 11 & 2 \\
\text{arc}(x_1, x_3), 16 & 0 & \text{arc}(x_2, x_3), 12 & 0 & \text{arc}(x_3, x_2), 9 & 0 \\
\text{arc}(x_1, x_4), 2 & 3 & \text{arc}(x_2, x_3), 21 & 0 & \text{arc}(x_3, x_2), 10 & 1 \\
\text{arc}(x_1, x_4), 10 & 1 & \text{arc}(x_2, x_4), 9 & 2 & \text{arc}(x_3, x_2), 11 & 1 \\
\end{array}
\]

\[ S[x_3, 9] = \{\langle x_1, 2 \rangle\} – \text{None-zero: no deletions} \]

Lecture 5: Global Arc Consistency
**AC4: Propagate**

\[ x_1, D_1 = \{2\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

**Counter**

<table>
<thead>
<tr>
<th>arc ((x_1, x_2)), 2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc ((x_1, x_2)), 10</td>
<td>0 arc ((x_2, x_1)), 9</td>
</tr>
<tr>
<td>arc ((x_1, x_2)), 16</td>
<td>0 arc ((x_2, x_1)), 12</td>
</tr>
<tr>
<td>arc ((x_1, x_3)), 2</td>
<td>2 arc ((x_2, x_1)), 21</td>
</tr>
<tr>
<td>arc ((x_1, x_3)), 10</td>
<td>1 arc ((x_2, x_3)), 9</td>
</tr>
<tr>
<td>arc ((x_1, x_3)), 16</td>
<td>0 arc ((x_2, x_3)), 12</td>
</tr>
<tr>
<td>arc ((x_1, x_4)), 2</td>
<td>3 arc ((x_2, x_3)), 21</td>
</tr>
<tr>
<td>arc ((x_1, x_4)), 10</td>
<td>1 arc ((x_2, x_4)), 9</td>
</tr>
</tbody>
</table>

\[ L = \{\langle x_4, 2 \rangle, \langle x_4, 5 \rangle, \langle x_4, 11 \rangle, \langle x_1, 10 \rangle\} \]
AC4: Propagate

\[ x_1, D_1 = \{2\} \quad x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \quad x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th>arc((x_1, x_2)), 2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc((x_1, x_2)), 10</td>
<td>0 arc((x_2, x_1)), 9</td>
</tr>
<tr>
<td>arc((x_1, x_2)), 16</td>
<td>0 arc((x_2, x_1)), 12</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 2</td>
<td>2 arc((x_2, x_1)), 21</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 10</td>
<td>1 arc((x_2, x_3)), 9</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 16</td>
<td>0 arc((x_2, x_3)), 12</td>
</tr>
<tr>
<td>arc((x_1, x_4)), 2</td>
<td>3 arc((x_2, x_3)), 21</td>
</tr>
<tr>
<td>arc((x_1, x_4)), 10</td>
<td>1 arc((x_2, x_4)), 9</td>
</tr>
</tbody>
</table>

\[ S[x_4, 2] = \{\langle x_3, 10\rangle, \langle x_3, 11\rangle\} \quad \text{– Decrement counters} \]
### AC4: Propagate

\[ x_1, D_1 = \{2\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{10, 11\} \]

\[ x_4, D_4 = \{10\} \]

#### Counter

<table>
<thead>
<tr>
<th>(\text{arc}(x_1, x_2)), 2</th>
<th>1</th>
<th>(\text{arc}(x_2, x_1)), 9</th>
<th>1</th>
<th>(\text{arc}(x_3, x_4)), 10</th>
<th>1</th>
<th>(\text{arc}(x_4, x_2)), 5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{arc}(x_1, x_2)), 10</td>
<td>0</td>
<td>(\text{arc}(x_2, x_1)), 12</td>
<td>2</td>
<td>(\text{arc}(x_3, x_1)), 9</td>
<td>1</td>
<td>(\text{arc}(x_3, x_4)), 11</td>
<td>2</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_2)), 16</td>
<td>0</td>
<td>(\text{arc}(x_2, x_1)), 21</td>
<td>2</td>
<td>(\text{arc}(x_3, x_1)), 10</td>
<td>1</td>
<td>(\text{arc}(x_4, x_1)), 2</td>
<td>0</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 2</td>
<td>2</td>
<td>(\text{arc}(x_2, x_1)), 9</td>
<td>2</td>
<td>(\text{arc}(x_3, x_1)), 11</td>
<td>2</td>
<td>(\text{arc}(x_4, x_1)), 5</td>
<td>1</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 10</td>
<td>1</td>
<td>(\text{arc}(x_2, x_3)), 9</td>
<td>2</td>
<td>(\text{arc}(x_3, x_1)), 10</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 10</td>
<td>1</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_3)), 16</td>
<td>0</td>
<td>(\text{arc}(x_2, x_3)), 12</td>
<td>0</td>
<td>(\text{arc}(x_3, x_2)), 9</td>
<td>0</td>
<td>(\text{arc}(x_4, x_1)), 10</td>
<td>1</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_4)), 2</td>
<td>3</td>
<td>(\text{arc}(x_2, x_3)), 21</td>
<td>0</td>
<td>(\text{arc}(x_3, x_2)), 10</td>
<td>1</td>
<td>(\text{arc}(x_4, x_1)), 11</td>
<td>2</td>
</tr>
<tr>
<td>(\text{arc}(x_1, x_4)), 10</td>
<td>1</td>
<td>(\text{arc}(x_2, x_4)), 9</td>
<td>2</td>
<td>(\text{arc}(x_3, x_2)), 11</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ S[x_4, 2] = \{\langle x_3, 10\rangle, \langle x_3, 11\rangle\} \] – None-zero: no deletions
AC4: Propagate

\( x_1, D_1 = \{2\} \)

\( x_2, D_2 = \{9\} \)

\( x_3, D_3 = \{10, 11\} \)

\( x_4, D_4 = \{10\} \)

\[ L = \{\langle x_4, 5 \rangle, \langle x_4, 11 \rangle, \langle x_1, 10 \rangle\} \]
AC4: Propagate

\[ x_1, D_1 = \{2\} \]

\[ x_2, D_2 = \{9\} \]

\[ x_3, D_3 = \{11\} \]

\[ x_4, D_4 = \{10\} \]

Counter

<table>
<thead>
<tr>
<th></th>
<th>arc((x_1, x_2)), 2</th>
<th>arc((x_1, x_2)), 10</th>
<th>arc((x_1, x_2)), 16</th>
<th>arc((x_1, x_3)), 2</th>
<th>arc((x_1, x_3)), 10</th>
<th>arc((x_1, x_3)), 16</th>
<th>arc((x_1, x_4)), 2</th>
<th>arc((x_1, x_4)), 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>arc((x_1, x_2)), 2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>arc((x_1, x_2)), 10</td>
<td>0</td>
<td>arc((x_2, x_1)), 9</td>
<td>1</td>
<td>arc((x_3, x_1)), 9</td>
<td>1</td>
<td>arc((x_3, x_4)), 10</td>
<td>0</td>
<td>arc((x_4, x_2)), 10</td>
</tr>
<tr>
<td>arc((x_1, x_2)), 16</td>
<td>0</td>
<td>arc((x_2, x_1)), 12</td>
<td>1</td>
<td>arc((x_3, x_1)), 10</td>
<td>1</td>
<td>arc((x_3, x_1)), 2</td>
<td>0</td>
<td>arc((x_4, x_2)), 11</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 2</td>
<td>1</td>
<td>arc((x_2, x_1)), 21</td>
<td>1</td>
<td>arc((x_3, x_1)), 11</td>
<td>1</td>
<td>arc((x_3, x_1)), 5</td>
<td>1</td>
<td>arc((x_4, x_2)), 11</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 10</td>
<td>1</td>
<td>arc((x_2, x_3)), 9</td>
<td>1</td>
<td>arc((x_3, x_1)), 11</td>
<td>1</td>
<td>arc((x_3, x_1)), 2</td>
<td>0</td>
<td>arc((x_4, x_2)), 11</td>
</tr>
<tr>
<td>arc((x_1, x_3)), 16</td>
<td>0</td>
<td>arc((x_2, x_3)), 12</td>
<td>0</td>
<td>arc((x_3, x_2)), 9</td>
<td>0</td>
<td>arc((x_4, x_1)), 10</td>
<td>1</td>
<td>arc((x_4, x_3)), 10</td>
</tr>
<tr>
<td>arc((x_1, x_4)), 2</td>
<td>1</td>
<td>arc((x_2, x_3)), 21</td>
<td>0</td>
<td>arc((x_3, x_2)), 10</td>
<td>1</td>
<td>arc((x_4, x_1)), 11</td>
<td>1</td>
<td>arc((x_4, x_3)), 11</td>
</tr>
<tr>
<td>arc((x_1, x_4)), 10</td>
<td>0</td>
<td>arc((x_2, x_4)), 9</td>
<td>1</td>
<td>arc((x_3, x_2)), 11</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Lecture 5: Global Arc Consistency

113
AC4: Time Complexity

- AC4 has worst-case time complexity of $O(ed^2)$.
  - $e$ is the number of edges in the constraint graph.
  - $d$ is the maximum domain size.
- Initialisation: $O(ed^2)$.
  - For each arc, double-nested loop of length $d$.
- Propagation: $O(ed^2)$.
  - At most $O(ed)$ counters, each decremented at most $d$ times.
- This is worst-case optimal.
AC4: Space Complexity

- AC4 has worst-case space complexity of $O(ed^2)$.
  - Initialisation: at most $ed^2$ additions to $S$.
    - So $S$ takes $O(ed^2)$ space.
  - The M array takes $O(nd)$ space.
  - The Counters array takes $O(ed)$ space.
- AC3 has worst-case space complexity of $O(e + nd)$.
  - This is optimal: require at least this much just to represent constraint graph and domains.
AC3 vs. AC4: Summary

• AC4 worst-case time is better than that of AC3.
  • But AC4 always reaches this worst case (initialization of data structures)
  • So people often prefer AC3 to AC4 in practice.
• AC3 has a better worst-case space complexity.
• Can we do better?
Lecture 5 Summary

- Enforcing Global Arc Consistency.
  - AC3 (Coarse-grained, sub-optimal).
  - AC4 (Fine-grained, optimal).
  - But in practice AC3 typically better than AC4!