Izmir University of Economics
Econ 533: Quantitative Methods and Econometrics

Constrained Optimization I

- The central mathematic problem in optimization is that of maximizing a function of several variables, where these variables are bound by constraining equations. The prototype problem is

$$\textit{maximize} \qquad f(x_1, ..., x_n)$$

where $(x_1, ..., x_n) \in \mathbf{R^n}$ must satisfy

$$g_1(x_1, ..., x_n) \leq b_1, ..., g_k(x_1, ..., x_n) \leq b_k,$$
$$h_1(x_1, ..., x_n) = c_1, ..., h_m(x_1, ..., x_n) = c_m.$$

Examples

- **Utility Maximization Problem**

$$\begin{aligned} &\textit{maximize} && U(x_1, ..., x_n) \\ &\textit{subject to} && p_1x_1 + p_2x_2 + ... + p_nx_n \leq I, \\ &&& x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0. \end{aligned}$$

- **Utility Maximization Problem with labor/Leisure Choice**

$$\begin{aligned} &\textit{maximize} && U(x_1, ..., x_n, l_1) \\ &\textit{subject to} && p_1x_1 + p_2x_2 + ... + p_nx_n \leq I' + wl_0, \\ &&& l_0 + l_1 = 24, \\ &&& x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0. \end{aligned}$$

- **Profit Maximization Problem of a Competitive Firm**

$$\textit{maximize} \qquad \Pi(x_1, ..., x_n) = pf(x_1, ..., x_n) - \sum_1^n w_i x_i$$

$$\begin{aligned} \textit{subject to} \qquad & pf(x_1, ..., x_n) - \sum_1^n w_i x_i \geq 0, \\ & g_1(\mathbf{x}) \leq b_1, ..., g_k(\mathbf{x}) \leq b_k, \\ & x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0. \end{aligned}$$

Two Variables and One Equality Constraint

$$\text{maximize} \qquad f(x_1, x_2)$$

$$\text{subject to} \qquad p_1 x_1 + p_2 x_2 = I$$

- Geometrically, the goal is to find the highest valued level curve of f which meets the constraint set.
- The highest level curve of f to touch the constraint set C must be *tangent* to C at the constrained max.
- The slope of the level set of f equals the slope of the constraint curve C at the constrained maximizer $\mathbf{x}^*$.

- The slope of the level set of f at x^*

$$-\frac{\partial f}{\partial x_1}(\mathbf{x}^*) \Big/ -\frac{\partial f}{\partial x_2}(\mathbf{x}^*) \tag{1}$$

- the slope of the constraint set $\{h(x_1, x_2) = c\}$ at $\mathbf{x}^*$ is

$$-\frac{\partial h}{\partial x_1}(\mathbf{x}^*) \Big/ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \tag{2}$$

- Two slopes are equal means

$$\frac{\frac{\partial f}{\partial x_1}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_1}(\mathbf{x}^*)} = \frac{\frac{\partial f}{\partial x_2}(\mathbf{x}^*)}{\frac{\partial h}{\partial x_2}(\mathbf{x}^*)} = \mu. \tag{3}$$

- Rewrite 3 as the two equations

$$\frac{\partial f}{\partial x_1}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_1}(\mathbf{x}^*) = 0, \tag{4}$$

$$\frac{\partial f}{\partial x_2}(\mathbf{x}^*) - \mu \frac{\partial h}{\partial x_2}(\mathbf{x}^*) = 0 \tag{5}$$

- Including the constraint equation $h(x_1, x_2) - c = 0$ with equations (4), (5) yields a system of three equations in three unknowns. Form the **Lagrangian function**

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu(h(x_1, x_2) - c). \tag{6}$$

- The new variable μ which multiplies the constraint is called a **Lagrange multiplier**.
- **Constraint qualification** $\partial h/\partial x_1$ or $\partial h/\partial x_2$ (or both) is not zero at the maximizer $\mathbf{x}^*$.

Theorem

Let f and h be C^1 functions of two variables. Suppose that $x^* = (x_1^*, x_2^*)$ is a solution to the problem

$$\begin{aligned} &maximize && f(x_1, x_2) \\ &subject\ \ to && h(x_1, x_2) = c \end{aligned}$$

Suppose futher that (x_1^*, x_2^*) is not a critical point of h. Then there is a real number μ such that (x_1^*, x_2^*, μ) is the critical point of the Lagrangian function

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu(h(x_1, x_2) - c).$$

In other words, at (x_1^*, x_2^*, μ)

$$\frac{\partial L}{\partial x_1} = 0, \qquad \frac{\partial L}{\partial x_2} = 0, \quad \text{and} \quad \frac{\partial L}{\partial \mu}.$$

Several Equality Constraints

$$\text{Maximize or minimize} \qquad f(x_1, ..., x_n).$$

$$\text{subject to} \qquad C_h = \{\mathbf{x} = (x_1, ..., x_n) : h_1(\mathbf{x}) = a_1, ..., h_m(\mathbf{x}) = a_m\}.$$

- If we have one constraint $h(x_1, ..., x_n) = a$, some first order partial derivative of h is not zero at the optimal $\mathbf{x}^*$.

$$\left(\frac{\partial h}{\partial x_1}(\mathbf{x}^*), \frac{\partial h}{\partial x_2}(\mathbf{x}^*), ..., \frac{\partial h}{\partial x_n}(\mathbf{x}^*)\right) \neq (0, 0, ..., 0) \qquad (7)$$

- If we have m constraint functions, m> 1, the natural generalization of (7) involves the Jacobian derivative

$$\mathbf{Dh}(\mathbf{x}^*) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \frac{\partial h_2}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_2}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

of the constraint functions. A point $\mathbf{x}^*$ is called a critical point of $h = (h_1, \dots, h_m)$ if the rank of the matrix $\mathbf{Dh}(\mathbf{x}^*)$ is $< m$. The natural generalization of the constraint qualification is that the rank of $\mathbf{Dh}(\mathbf{x}^*)$ be m - as large as it can be. **Formally,** $(h_1, \dots, h_m)$ satisfies the nondegenerate constraint qualification (NCDQ) at $\mathbf{x}^*$ if the rank of $\mathbf{Dh}(\mathbf{x}^*)$ at $\mathbf{x}^*$ is m.

Theorem

Let f and $h_1, ..., h_m$ be C^1 functions of n variables. Consider the problem of maximizing (or minimizing) $f(\mathbf{x})$ on the constraint set

$$c_h \equiv \{x = (x_1, ..., x_n) : h_1(x) = a_1, ..., h_m(x) = a_m\}.$$

Suppose that $\mathbf{x}^* \in C_h$ and that $\mathbf{x}^*$ is a local max or min of f on C_h. Suppose further that x^* satisfies condition NDCQ. Then, there exist $\mu_1^*, ..., \mu_m^*$ such that $(x_1^*, ..., x_n^*, \mu_1^*, ..., \mu_m^*) \equiv (\mathbf{x}^*, \ \mu^*)$ is a critical point of the Lagrangian

$$L(\mathbf{x}, \mu) = f(\mathbf{x}) - \mu_1[h_1(\mathbf{x}) - a_1] - \mu_2[h_2(\mathbf{x}) - a_2] - .... - \mu_m[h_m(\mathbf{x}) - a_m].$$

In other words,

$$\begin{aligned} \frac{\partial L}{\partial x_1}(\mathbf{x}^*, \mu^*) &= 0, ..., \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \mu^*) = 0, \\ \frac{\partial L}{\partial \mu_1}(\mathbf{x}^*, \mu^*) &= 0, ..., \frac{\partial L}{\partial \mu_m}(\mathbf{x}^*, \mu^*) = 0. \end{aligned}$$

One Inequality Constraint

$$\begin{array}{ll} \text{Maximize} & f(x,y). \\ \text{subject to} & g(x,y) \leq b. \end{array}$$

- Form the Lagrangian

$$L(x,y,\lambda) = f(x,y) - \lambda[g(x,y) - b]$$

- The constraint is binding (or is active, effective or tight), that is, $g(x,y) - b = 0$, in which case λ must be ≥ 0, or the constraint is not binding in which case λ must be zero.
- **Complemantary Slackness Condition**

$$\lambda.[g(x,y) - b] = 0.$$

Theorem

Suppose that f and g are C^1 functions on $\mathbf{R}^2$ and that (x^*, y^*) maximizes f on the constraint set $g(x, y) \leq b$. If $g(x^*, y^*) = b$, suppose that

$$\frac{\partial g}{\partial x}(x^*, y^*) \neq 0 \quad \text{or} \quad \frac{\partial g}{\partial y}(x^*, y^*) \neq 0$$

In any case form the Lagrangian function

$$L(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - b].$$

Then there is a multiplier λ^* such that:

1. $\frac{\partial L}{\partial x}(x^*, y^*, \lambda^*) = 0$,
2. $\frac{\partial L}{\partial y}(x^*, y^*, \lambda^*) = 0$,
3. $\lambda.[g(x, y) - b] = 0$.
4. $\lambda^* \geq 0$,
5. $g(x^*, y^*) \leq b$

Notice the similarities and differences between the statatements of the theorem regarding equality constraints and the theorem which covers inequality constraints:

1. Both use the **same** Lagrangian L and both require the derivatives of L with x_i's be zero.
2. The condition that $\partial L/\partial\mu = h(x,y) - c = 0$ for equality constraints may no longer hold for inequality constraints since the constraint need not be binding at the maximizer in the inequality constraint case. It is replaced by two conditions:

$$\lambda.[g(x,y) - b] = 0. \quad \textit{and} \quad \frac{\partial L}{\partial \lambda} = g(x,y) - b \leq 0.$$

3. We check a constraint qualification for both situations. We need only to check a constraint qualification for an inequality constraint if that constraint is binding at the solution candidate.
4. There are no restrictions on the sign of the multiplier in the equality constraint situation; however the multiplier for inequality constraints must be nonnegative.

Several Inequality Constraints

Theorem

Suppose that $f, g_1, \ldots, g_k$ are C^1 functions of n variables. Suppose that $\mathbf{x}^* \in \mathbf{R^n}$ is a local maximizer of f on the constraint set defined by k inequalities

$$g_1(x_1, \ldots, x_n,) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k.$$

Assume that the first k_0 constraints are binding at $\mathbf{x}^*$ and that the last $k - k_0$ constraints are not binding. Suppose that the following NCDQ is satisfied at $\mathbf{x}^*$.
The rank at $\mathbf{x}^*$ of the Jacobian matrix of the binding constraints

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

is k_0 - as large as it can be.
Form the Lagrangian

$$L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k) \equiv f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \ldots - \lambda_k[g_k(\mathbf{x}) - b_k].$$

Then, there exist multipliers $\lambda_1^*, \ldots, \lambda_l^*$ such that:

1. $\frac{\partial L}{\partial x_1}(\mathbf{x}^*, \lambda^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \lambda^*) = 0$
2. $\lambda_1^*.[g_1(\mathbf{x}) - b_1] = 0, \ldots, \lambda_k^*.[g_k(\mathbf{x}) - b_k] = 0$
3. $\lambda_1^* \geq 0, \ldots, \lambda_k^* \geq 0,$
4. $g_1(\mathbf{x}^*) \leq b_1, \ldots, g_k(\mathbf{x}^*) \leq b_k.$

Mixed Constraints

Theorem

Suppose that $f, g_1, \ldots, g_k, h_1, \ldots, h_m,$ *are* C^1 *functions of* n *variables. Suppose that* $\mathbf{x}^* \in \mathbf{R^n}$ *is a local maximizer of* f *on the constraint set defined by* k *inequalities and* m *equalities.*

$$g_1(x_1, \ldots, x_n,) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k.$$

$$h_1(x_1, \ldots, x_n,) = c_1, \ldots, h_m(x_1, \ldots, x_n) = c_m.$$

Assume that the first k_0 *constraints are binding at* $\mathbf{x}^*$ *and that the last* $k - k_0$ *constraints are not binding. Suppose that the following NCDQ is satisfied at* $\mathbf{x}^*$*: The rank at* $\mathbf{x}^*$ *of the Jacobian matrix of the equality constraints and the binding inequality constraints*

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}^*) \\ \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

is $k_0 + m$ - as large as it can be. Form the Lagrangian

$$\begin{aligned} & L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_m) \\ & \qquad \equiv f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \dots - \lambda_k[g_k(\mathbf{x}) - b_k] \\ & \qquad - \mu_1[h_1(\mathbf{x}) - c_1] - \dots - \mu_m[h_m(\mathbf{x}) - c_m]. \end{aligned}$$

Then, there exist multipliers $\lambda_1^*, \ldots, \lambda_k^*, \mu_1^*, \ldots, \mu_m^*$ such that:

1. $\frac{\partial L}{\partial x_1}(\mathbf{x}^*, \lambda^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \lambda^*) = 0$
2. $\lambda_1^*.[g_1(\mathbf{x}) - b_1] = 0, \ldots, \lambda_k^*.[g_k(\mathbf{x}) - b_k] = 0$
3. $h_1(\mathbf{x}) = c_1, \ldots, h_m(\mathbf{x}) = c_m$,
4. $\lambda_1^* \geq 0, \ldots, \lambda_k^* \geq 0$,
5. $g_1(\mathbf{x}^*) \leq b_1, \ldots, g_k(\mathbf{x}^*) \leq b_k$.

Constrained Minimization Problems

Theorem

*Suppose that $f, g_1, \ldots, g_k, h_1, \ldots, h_m$, are C^1 functions of n variables. Suppose that $\mathbf{x}^* \in \mathbf{R^n}$ is a local **minimizer** of f on the constraint set defined by k inequalities and m equalities.*

$$g_1(x_1, \ldots, x_n,) \geq b_1, \ldots, g_k(x_1, \ldots, x_n) \geq b_k.$$

$$h_1(x_1, \ldots, x_n,) = c_1, \ldots, h_m(x_1, \ldots, x_n) = c_m.$$

Assume that the first k_0 constraints are binding at $\mathbf{x}^*$ and that the last $k - k_0$ constraints are not binding. Suppose that the following NCDQ is satisfied at $\mathbf{x}^*$: The rank at $\mathbf{x}^*$ of the Jacobian matrix of the binding constraints

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(\mathbf{x}^*) \\ \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \cdots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

is $k_0 + m$ - as large as it can be. Form the Lagrangian

$$\begin{aligned} & L(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_m) \\ & \qquad \equiv f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \ldots - \lambda_k[g_k(\mathbf{x}) - b_k] \\ & \qquad - \mu_1[h_1(\mathbf{x}) - c_1] - \ldots - \mu_m[h_m(\mathbf{x}) - c_m]. \end{aligned}$$

Then, there exist multipliers $\lambda_1^*, \ldots, \lambda_k^*, \mu_1^*, \ldots, \mu_m^*$ such that:

1. $\frac{\partial L}{\partial x_1}(\mathbf{x}^*, \lambda^*) = 0, \ldots, \frac{\partial L}{\partial x_n}(\mathbf{x}^*, \lambda^*) = 0$
2. $\lambda_1^*.[g_1(\mathbf{x}) - b_1] = 0, \ldots, \lambda_k^*.[g_k(\mathbf{x}) - b_k] = 0$
3. $h_1(\mathbf{x}) = c_1, \ldots, h_m(\mathbf{x}) = c_m,$
4. $\lambda_1^* \geq 0, \ldots, \lambda_k^* \geq 0,$
5. $g_1(\mathbf{x}^*) \geq b_1, \ldots, g_k(\mathbf{x}^*) \geq b_k.$

A recipe for the Lagrange Method

- Formulate $g(x) \leq b (g(x) \geq b)$ for maximization (minimization) problems
- Check the constraint qualification
- Introduce the Lagrange multiplier(s) and set up the Lagrangian
- Write the first order conditions (FOCs)
- Determine the critical points satisfying the FOCs
- Evalauate the objective function at each critical point

Kuhn-Tucker Formulation

- The most common maximization problems in economics involve only inequality constraints and nonnegativity constraints.

$$\begin{aligned} \text{maximize} \quad & f(x_1, \ldots, x_n) \\ \text{subject to} \quad & g_1(x_1, \ldots, x_n,) \leq b_1, \ldots, g_k(x_1, \ldots, x_n) \leq b_k. \\ & x_1 \geq 0, \ldots, x_n \geq 0. \end{aligned} \tag{8}$$

- We would write the Lagrangian as

$$\begin{aligned} & L(\mathbf{x}, \lambda_1, \ldots, \lambda_k, v_1, \ldots, v_n) \\ & \quad = f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \ldots - \lambda_k[g_k(\mathbf{x}) - b_k] + v_1 x_1 + \ldots, v_n x_n. \end{aligned}$$

- The FOCs are

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \ldots - \lambda_k \frac{\partial g_k}{\partial x_1} + v_1 &= 0, \\
\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda_n \frac{\partial g_1}{\partial x_n} - \ldots - \lambda_k \frac{\partial g_k}{\partial x_n} + v_n &= 0, && (9) \\
\lambda_1(g_1(\mathbf{x}) - b_1) = -\lambda_1 \frac{\partial L}{\partial \lambda_1} &= 0, \\
\lambda_k(g_k(\mathbf{x}) - b_k) = -\lambda_k \frac{\partial L}{\partial \lambda_k} &= 0, && (10) \\
v_1 x_1 &= 0, \\
v_n x_n &= 0, && (11) \\
\lambda_1, \ldots, \lambda_k, v_1, \ldots, v_n &\geq 0,
\end{aligned}
$$

plus the inequalities in (8).

- $\tilde{L}$, Kuhn-Tucker Lagrangian, does not include nonnegativity constraints:

$$\tilde{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_k) \equiv f(\mathbf{x}) - \lambda_1[g_1(\mathbf{x}) - b_1] - \ldots - \lambda_k[g_k(\mathbf{x}) - b_k] \quad (12)$$

- Note that

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_k, v_1, \ldots, v_n) = \tilde{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_k) + v_1 x_1 + \ldots + v_n x_n.$$

- For $j = 1, \ldots, n$, write (9) as

$$\begin{aligned} \frac{\partial L}{\partial x_j} &= \frac{\partial \tilde{L}}{\partial x_j} + v_j = 0, \\ \frac{\partial \tilde{L}}{\partial x_j} &= -v_j \end{aligned} \quad (13)$$

at the solution for each j.

- By (11), (13), and the equations $v_j \geq 0$,

$$\frac{\partial \tilde{L}}{\partial x_j} \leq 0, x_j \frac{\partial \tilde{L}}{\partial x_j} = 0. \tag{14}$$

- On the other hand, for any $\mathbf{x}$,

$$\frac{\partial \tilde{L}}{\partial \lambda_j} = \frac{\partial L}{\partial \lambda_j} = b_j - g_j(\mathbf{x}) \geq 0. \tag{15}$$

- Combining (10), (14), and (15) the first order conditions in terms of the Kuhn-Tucker Lagrangian are

$$\frac{\partial \tilde{L}}{\partial x_1} \leq 0, \ldots, \frac{\partial \tilde{L}}{\partial x_n} \leq 0, \qquad \frac{\partial \tilde{L}}{\partial \lambda_1} \geq 0, \ldots, \frac{\partial \tilde{L}}{\partial \lambda_k} \geq 0,$$

$$x_1 \frac{\partial \tilde{L}}{\partial x_1} = 0, \ldots, x_n \frac{\partial \tilde{L}}{\partial x_n} = 0, \qquad \lambda_1 \frac{\partial \tilde{L}}{\partial \lambda_1} = 0, \ldots, \lambda_k \frac{\partial \tilde{L}}{\partial \lambda_k} = 0,$$

- Advantages of Kuhn-Tucker formulation
  1. It involves $n+k$ equations in $n+k$ unknowns, compared with the $2n+k$ equations in $2n+k$ unknowns.
  2. x_i's and λ_j's enter the first order conditions in a symmetric way.