Izmir University of Economics
Econ 533: Quantitative Methods and Econometrics

Multivariable Optimization

Introduction

- Most interesting economic optimization problems require the simultaneous choice of several variables.
- For instance, a profit maximizer producer of a single commodity chooses quantities of different inputs.
- A consumer chooses what quantities of the many different goods to buy.
- We begin by two variables and present the basic results.
- Then we give a more systematic presentation of the theory with two variables.

Two Variables: Necessary Conditions

A point (x_0, y_0) where both partial derivatives are 0 is called a **stationary point** of f.

Theorem (Necessary Conditions for Interior Extrama)

A differentiable function $z = f(x, y)$ can only have a maximum or minimum at an interior point (x_0, y_0) of S if it is a stationary point-that is, if the point $(x, y) = (x_0, y_0)$ satisfies the two equations $f_1'(x, y) = 0$, $f_2'(x, y) = 0$ ***(first order conditions, or FOCs)***

Examples

- The function f is defined for all (x,y) by

$$f(x,y) = -2x^2 + -2xy - 2y^2 + 36x + 42y - 158$$

Assume f has a maximum point. Find it.

- A firm produces two different kinds of A and B of a commodity. The daily cost of producing x units of A and y units of B is

$$C(x,y) = 0.04x^2 + 0.01xy + 0.01y^2 + 4x + 2y + 500$$

Suppose the firm sells all its output at a price per unit of 15 for A and 9 for B. Find the daily production levels x and y that maximize profit per day.

Examples

- Suppose that $Q = F(K, L)$ is a production function with K as the capital input and L as the labor input. The price per unit of output is p, the cost of capital is r, and the wage rate is w. The constants p, r, and w are all positive. Write the first order conditions. And interpret them.
- Find the only solution to the following special case of the previous Example.

$$max\pi(K, L) = 12K^{1/2}L^{1/4} - 1.2K - 0.6L$$

Two Variables: Sufficient Conditions

- Suppose f is a function of one variable which is twice differentiable in an interval I. A sufficient condition for a stationary point in I to be a maximum point is that $f''(x) \leq 0$ for all x in I. The function f is then called concave.
- A set S in the xy plane is convex if, for each pair of points P and Q in S, all the line segment between P and Q lies in S.

Two Variables: Sufficient Conditions

Theorem (Sufficient Conditions for a Maximum and Minimum)

Suppose that (x_0, y_0) is a stationary point for a C^2-function $f(x, y)$ in a convex set S.

a. *If for all (x, y) in S,*
   $f''_{11}(x, y) \leq 0, f''_{22}(x, y) \leq 0$, *and*
   $f''_{11}(x, y)f''_{22}(x, y) - (f''_{12}(x, y))^2 \geq 0$ *then* (x_0, y_0)*is a maximum point for* (x, y) *in* S.

b. *If for all (x, y) in S,*
   $f''_{11}(x, y) \geq 0, f''_{22}(x, y) \geq 0$, *and*
   $f''_{11}(x, y)f''_{22}(x, y) - (f''_{12}(x, y))^2 \geq 0$ *then* (x_0, y_0)*is a minimum point for* (x, y) *in* S.

Examples

- Show that $f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$ has a maximum at the stationary point $(x_0, y_0) = (5, 8)$.
- Show that we found the maximum in the example $\pi(K, L) = 12K^{1/2}L^{1/4} - 1.2K - 0.6L$
- Suppose that any production by the firm in Example 1 creates pollution, so it is legally restricted to produce a total of 320 units of the two kinds of output. The firm's problem is then: max $-0.04x^2 - 0.01xy - 0.01y^2 + 11x + 7y - 500$ subject to $x + y = 320$. What are the optimal quantities of the two kinds of output?

Local Extreme Points

- The point (x_0, y_0) is said to be a local maximum point of f in S in $f(x, y) \leq f(x_0, y_0)$ for all pairs (x, y) in S that lie sufficiently close to (x_0, y_0). If the inequality is strict for $(x, y) \neq (x_0, y_0)$, then (x_0, y_0) is a **strict** local maximum point.
- Global extreme point is also a local extreme point; the converse is not true.
- At a local extreme point in the interior of the domain of a differentiable function, all first order partial derivatives are 0.
- The first order conditions are necessary for a differentiable function to have a local extreme point. However a stationary point does not have to be a local extreme point. A saddle point is neither a local maximum nor a local minimum.

Local Extreme Points

- A saddle point (x_0, y_0) is a stationary point with the property that there exists points (x, y) arbitrarily close to (x_0, y_0) with $f(x, y) \leq f(x_0, y_0)$, and there also exists such points with $f(x, y) \geq f(x_0, y_0)$.
- Example: Show that (0,0) is a saddle point of $f(x, y) = x^2 - y^2$.
- The stationary points of a function thus fall into three categories.
  (a) Local maximum points
  (b) Local minimum points
  (c) Saddle points
- To help decide whether a given point is of type a, b, or c, one can use the second derivative test.

Second Derivative Test

Theorem (Second derivative test for local extrema)

Suppose $f(x, y)$ is a function with continuous second order partial derivatives in a domain S, and let (x_0, y_0) be an interior stationary point of S. Write
$A = f''_{11}(x_0, y_0)$, $B = f''_{12}(x_0, y_0)$, *and* $C = f''_{22}(x_0, y_0)$

(i) *If $A < 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a (strict) local maximum point.*

(ii) *If $A > 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a (strict) local minimum point.*

(iii) *If $AC - B^2 < 0$, then (x_0, y_0) is a saddle point.*

(iv) *If $AC - B^2 = 0$, then (x_0, y_0) could be a local maximum, a local minimum, or a saddle point.*

Examples

- Find the stationary points and classify them when $f(x,y)x^3 - x^2 - y^2 + 8$.
- Consider the profit maximization example and suppose that F is twice differentiable. Let (K^*, L^*) be an input pair satisfying the first order conditions. Prove that (K^*, L^*) is a local maximum for the profit function.

Linear Models with quadratic objectives

- **Example 1 (Discriminating Monopolist)** Consider a firm that sells a product in two isolated geographical areas.If it wants to, it can charge different prices in the two different areas because what is sold in one area cannot be resold in the another. Suppose that such a firm has some monopoly power to influence the different prices it faces in the two seperate markets by adjusting the quantity it sells in each. Economists generally use the term "‘discriminating monopolist"’ to describe a firm having this power. Faced with two such isolated markets, the discriminating monopolist has two independent demand curves. Suppose that, in inverse form, these are

$$\begin{aligned} P_1 &= a_1 - b_1 Q_1 \\ P_2 &= a_2 - b_2 Q_2 \end{aligned}$$

for markets 1 and 2, respectively.

Linear Models with quadratic objectives

- Suppose, too, that the total cost is proportional to total production:

$$C(Q) = \alpha(Q_1 + Q_2)$$

  Find the values of Q_1 and Q_2 that maximize profits.

- **Example 2:** Suppose the monopolist in Example 1 has the demand functions

$$\begin{aligned} P_1 &= 100 - Q_1 \\ P_2 &= 80 - Q_2 \end{aligned}$$

  and that the cost function is $C = 6(Q_1 + Q_2)$.
  (a) How much should be sold in the two markets to maximize profits? What are the corresponding prices?
  (b) How much profit is lost if it becomes illegal to discriminate?
  (c) The authoriities in market 1 impose a tax of t per unit sold in market 1. Discuss the consequences.

Linear Models with quadratic objectives

- **Example 3 (Discriminating Monopsonist)** Consider a firm using quantities L_1 and L_2 of two kinds of labor as its only inputs in order to produce output Q according to the simple production function $Q = L_1 + L_2$. Thus, both output and labor supply are measured so that each unit of labor produces one unit of output. Note especially how the two kinds of labor essentially distinguishable because each unit of each type makes equal contribution to the firm's level of output. Suppose, howver, that there are two segmented labor markets, with different inverse supply functions specifying the wage that must be paid to attract a given labor supply. Specifically, suppose that

$$w_1 = \alpha_1 + \beta_1 L_1, \quad w_2 = \alpha_2 + \beta_2 L_2$$

Assume moreover that the firm is competitive in its output market, taking price P as fixed. Find the values of L_1 and L_2 that maximize firm's profits.

The Extreme Value Theorem

- A point (a,b)is called an interior point of a set S in the plane if there exists a circle centred at (a, b) such that all points strictly inside the circle lie in S.
- A set is called open if it consists only of interior points.
- The point (a, b) is called a boundary point of a set S if every circle centered at (a, b) contains points of S as well as points in its complement.
- If S contains all its boundary points, then S is called **closed**.
- A set that contains some but not all its boundary point is neither open nor closed.
- A set in the plane is **bounded** if the whole set is contained within a sufficiently large circle.
- A set in the plane that is both closed and bounded is called **compact**.

The Extreme Value Theorem

Theorem (Extreme Value Theorem)

Suppose the function $f(x, y)$ is continuous throughout a nonempty, closed and bounded set S in the plane. Then there exists both a point (a, b) in S where f has a minimum and a point in S where f has a maximum- that is, $f(a, b) \leq f(x, y) \leq f(c, d)$ for all (x, y) in S

Finding Maxima and Minima

- Find the maximum and minimum values of a differentiable function $f(x, y)$ defined on a closed, bounded set S in the plane.
- Solution:

(I) Find all stationary points of f in the interior of S.

(II) Find the largest value and the smallest value of f on the boundary of S, along with the associated points. (If it is convenient to subdivide the boundary into several pieces, find the largest and the smallest value on each piece of the boundary.)

(III) Compute the values of the function at all points found in (I) and (II). The largest function at all points found in (I) and (II) is the maximum value of f in S. The smallest function value is the maximum value of f in S.

Examples

- **Example 1:** Find the extreme values for $f(x, y)$ defined over S when

$$f(x, y) = x^2 + y^2 + y - 1, \quad S = \{(x, y) : x^2 + y^2 \leq 1\}$$