Izmir University of Economics
Econ 533: Quantitative Methods and Econometrics

The Time Value of Money

Introduction

- The main concern is how the values of investments and loans at different times are affected by interest rates.
- This lecture discusses effective rates of interest, continuously compounded interest, present values of future claims, annuities, and the internal rate of return on investment projects.

Interest periods and effective rates

- Suppose a principal of S_0 yields interest at the rate p percent per period. After t periods it will have increased to the amount
$$S(t) = S_0(1+r)^t$$
where $r = p/100$
- Each period the principal increases by the factor $1+r$. Note that p percent means $p/100$, and we say the interest rate is $p\%$ or r.
- If the interest is paid biannually at the rate of $p/2\%$. Then the principal after half a year it will have increased to
$$S_0 + S_0\frac{p/2}{100} = S_0(1+\frac{r}{2})$$

Interest periods and effective rates

- Each half year the principal increases by the factor $1 + r/2$. After 2 periods it will have increased to $S_0\left(1+\frac{r}{2}\right)^2$, and after t periods it will have increased to $S_0\left(1+\frac{r}{2}\right)^{2t}$
- Note that a biannual interest payment is better than an annual interest payment because $(1 + r/2)^2 = 1 + r + r^2/4 > 1 + r$.
- n=4 if interest is paid quarterly, n=12 if it is paid monthly.
- More generally

$$S_0\left(1+\frac{r}{n}\right)^{nt}$$

- The greater is n, the faster interest accrues to the lender.

Examples

- Example 1: A deposit of 5 000 TL is put into an account earning interest at the annual rate of 9 percent, with interest paid quarterly. How much will there be in the account after 8 years?
- Example 2: How long will it take for the 5 000 TL to increase to 15 000 TL?

Effective Rate of Interest

- Effective interest rate is used in comparing various offers.
- Consider a loan giving an interest rate of 9 percent with interest at the rate 9/12=0.75 percent added twelve times a year.
- Initial principal will have grown to a debt of $S_0(1+0.09/12)^{12} \approx S_0 1.094$.
- The debt will grow at a constant proportional rate of 9.4 percent per year
- **Effective Yearly Rate** When interest is added n times during the year at the rate r/n per period, then the effective yearly rate R is defined as
$$R = \left(1+\frac{r}{n}\right)^n - 1$$
- The effective yearly rate is independent of the amount S_0. For a given value of $r > 0$, it is increasing in n.

Examples

- Example 1: What is effectively yearly rate R corresponding to an annual interest rate of 9 percent with interest compounded (i) each quarter; (ii) each month?
- Example 2: When investing in a savings account, which of the following offers are better: 5.9 percent with interest paid quarterly; or 6 percent with interest paid twice a year?

Continuous Compounding

- What happens as n tends to infinity? We put $r/n = 1/m$. Then, $n = mr$ and so

$$S_0\left(1+\frac{r}{n}\right)^{nt} = S_0\left(1+\frac{1}{m}\right)^{mrt} = S_0\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \quad (1)$$

- As $n \Rightarrow \infty$ (with r fixed), so $m = n/r \Rightarrow \infty$, and we have $(1+1/m)^m \Rightarrow e$.
- The expression in (1) tends to $S_0 e^{rt}$ as n tends to infinity, implying that interest is compounded more and more frequently. In the limit, we talk about continuous compounding of interest.

Continuous Compounding

Continuous Compounding of Interest

The formula

$$S(t) = S_0 e^{rt}$$

shows how much a principal of S_0 will have increased to after t years, if the annual interest is r, and there is continuous compounding of interest.

Examples

- Example 1: Suppose the sum of 5 000 TL is invested in an account earning interest at an annual rate of 9 percent. What is the balance after 8 years if interest is compounded continuously?
- Example 2: Find the amount K by which 1 dollar increase in the course of a year when the interest rate is 8 percent per yearand interest is added: (i) yearly; (ii) biannually; (iii) continuously.

Present Value

Present Discounted Value:
If the interest or discount rate is p percent per year and $r = p/100$, an amount K that is payable in t years has the present value (or present discounted value, or PDV):

- $K(1+r)^{-t}$, with annual interest payments
- Ke^{-rt}, with continous compounding of interest

Examples

- Example 1: Find the present value of 100 000 TL which is due for payment after 15 years if the interest rate is 6 percent per year, compunded (i) annually or (ii) continuously.
- Example 2: Consider a tree that is planted at time $t = 0$, and let $P(t)$ be its current market value at time t, where $P(t)$ is differentiable. Assume that the interest rate is is 100r percent per year, and assume continuous compounding of interest.
    a. At what time t^* should this tree be cut down in order to maximize its present value?
    b. The optimal cutting time t^* depends on the interest rate r. Find dt^*/dr.

Geometric Series

- Example: This year a firm has annual revenue of 100 million TL that it expects to increase by 16 percent per year throughout the decade. How large is its expected revenue in the tenth year, and what is the total revenue expected over the whole period?
- Solution: $100 + 100 * 1.16 + 100 * (1.16)^2 + ... + 100 * (1.16)^9$
- In general we want to find the sum of

$$s_n = a + ak + ak^2 + ... + ak^{n-2} + ak^{n-1} \tag{2}$$

- We call this a sum of geometric series with quotient k.
- First multiply both sides of by k to obtain

$$ks_n = ak + ak^2 + ak^3 + ... + ak^{n-1} + ak^n$$

- Subtracting (2) from this equation yields

$$ks_n - s_n = ak^n - a \tag{3}$$

Geometric Series

- If $k = 1$, the sum is equal to $s_n = an$.
- For $k \neq 1$, implies that

$$s_n = a\frac{k^n - 1}{k - 1}$$

- **Summation formula for a finite geometric series**

$$ak + ak^2 + ak^3 + ... + ak^{n-1} = a\frac{k^n - 1}{k - 1} \qquad (k \neq 1) \qquad (4)$$

- Example: Find the sum in the example. Notice that we have $a = 100$, $k = 1.16$, and $n = 10$.

Infinite Geometric Series

- Consider the infinite sequence of numbers

$$1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}} \tag{5}$$

- What is meant by the infinite sum?

$$1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} + \dots$$

- As n increases, the term $1/2^{n-1}$ comes closer and closer to 0, and the sum in (5) tends to 2 as limit. This makes it natural to define the infinite sum as the number 2.

Infinite Geometric Series

- What meaning can be given to the infinite sum?

$$a + ak + ak^2 + ... + ak^{n-1} + ... \tag{6}$$

- We use the same idea as in (7), consider the sum s_n of the n first terms in (8). According to (4)

$$s_n = a\frac{1-k^n}{1-k} \quad (k \neq 1) \tag{7}$$

- What happens to this expression as n tends to infinity depends on k^n. In fact, k^n tends to 0 if $-1 < k < 1$, whereas k^n does not tend to any limit if $k > 1$ or $k \leq -1$.

Infinite Geometric Series

Formula for an infinite geometric series

$$a + ak + ak^2 + ... + ak^{n-1} + ... = \frac{a}{1-k} \quad \textit{if } |k| < 1$$

- If $|k| < 1$, the sum tends to the limit $a/(1-k)$ as n tends to infinity and the infinite series converges.
- If $|k| \geq 1$, the series diverges. A divergent series has no infinite sum.
- When $k = 1$, $s_n = na$, which tends to $+\infty$ if $a > 0$, and tends to $-\infty$ if $a < 0$.
- When $k = -1$, $k^n = -1$ and $s_n = a$, when n is odd, but $k^n = 1$ and $s_n = 0$, when n is even; again there is no limit as $n \to \infty$.

Infinite Geometric Series

Example : A rough estimate of the total oil and gas reserves under the Norwegian continental shelf at the beginning of 1999 was 13 billion (13.10^9) tons. Output that year was approximately 250 million (250.10^6) tons.

a. When will the reserves be exhausted if output is kept at the same constant level?

b. Suppose that output is reduced each year by 2 percent per year beginning in 1999. How long will the reserves last in this case?

Total Present Value

- Suppose that n successive payments $a_1, a_2, ..., a_n$ are to be made, with a_1 being paid after 1 year, a_2 after 2 years, and so on.
- How much must be deposited into an account today in order to have enough savings today to cover all these future payments, given that the annual interest rate is r?
- In other words, what is the present value of all these payments?
- The total amount P_n that must be deposited today in order to cover all n payments is therefore

$$\frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n}$$

- Here, P_n is the present value of the n installements.

Total Present Value

- An annuity is a sequence of equal payments made at fixed periods of time over some time span.
- If $a_1 = a_2 = ...a_n = a$ then represents the present value of annuity.
- In this case, the sum is a finite geometric series with n terms. The first term is $a/(1+r)$ and $k = 1/1+r$.
- According to the summation formula, the sum is

$$P_n = \frac{a}{1+r}\frac{[1-(1+r)^{-n}]}{[1-(1+r)^{-1}]} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

Present Value of an Annuity

The present value of an annuity of a per payment period for n periods at the rate of interest r per period, where each payment is at the end of the period, is given by

$$P_n = \frac{a}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n} = \frac{a}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

Future Value of an Annuity

- If we want to find how much has accumulated in the account after n periods, then the future value F_n of the annuity is given by

$$F_n = a + a(1+r) + a(1+r)^2 + ... + a(1+r)^{n-1}$$

- The summation formula for a geometric series yields

$$F_n = \frac{a[1-(1+r)^n]}{1-(1+r)} = \frac{a}{r}\left[(1+r)^n - 1\right]$$

Future Value of an Annuity

An amount a is deposited in account each period for n periods, earning interest at r per period. The future (total) value of the account, immediately after the last deposit, is

$$F_n = \frac{a}{r}\left[(1+r)^n - 1\right]$$

Examples

- Example 1: Compute the present and future value of a deposit of 1000 dollars in each of the coming 8 years if the annual interst rate is 6 percent.
- Example 2: Compute the present value of a series of deposits of 1000 dollars at the end of each year in perpetuity when the annual interest rate is 14 percent.

Mortgage Payments

- Like annuity, equal payments are due each period for a home *mortgage* at a fixed interest rate.
- Each payment goes partly to pay interest on the outstanding principal, and partly to repay principal.
- The interest part is largest in the beginning, because interest has to be paid on the whole loan for the *first period*.
- The interest part is smallest in the *last period*, because by then the outstanding balance is small.
- For the principal payment - the difference between the fixed monthly payment and the interest- it is the other way around.

Examples

- Example 1: A person borrows 50 000 TL at the beginning of a year and is supposed to pay it off in equal installements at the end of each year, with interest at 15 percent compounding annually. Find the annual payment.

Year	Payment	Interest	Principal Repay.	Outs. Balance
1	14 915.78	7 500.00	7 415.78	42 584.22
2	14 915.78	6 387.63	8 258.15	34 056.07
3	14 915.78	5 108.41	9 807.37	24 248.70
4	14 915.78	3 637.31	11 278.41	12 970.23
5	14 915.78	1 945.55	12 970.23	0

Examples

- Example 2: Suppose that the loan in Example 1 is being repaid by monthly payments at the end of each month with interst at the nominal rate 15 percent per year compounding monthly. Find the monthly payment.

Annuity Due

- *The annuities considered so far were* ***ordinary*** *annuities where each payment is made at the end of the period. If the payment each period is made at the beginning of the period, the annuity is called an* ***annuity due****. This kind of annuity can be regarded as ordinary annuity, except that there is an immediate initial payment.*
- Example 3: A person is assuming responsibility for a 335 000 TL loan which should be repaid in 15 equal repayments of a TL, the first one immediately and the following after each of the coming 14 years. Find a if the annual interest rate is 14 percent.

Fixed payments

- Some lenders prefer to specify a fixed payment each period. The difference is that there will be a final adjustment in the last payment in order for the present value of all the payments to be equal to the borrowed amount.

The number of periods n needed to pay off a loan of amount K at the rate a per period, when the interest rate is r per period, is given by

$$n = \frac{ln a - ln(a - rK)}{ln(1 + r)}$$

Examples

- Example 4: A loan of 50 000 TL is to be repaid by paying 20 000 TL, which covers both interest and the principal repayment, at the end of each of the coming years, until the loan is fully paid off. When is the loan paid off, and what is the final payment if the annual interest rate is 15 percent?

Deposits within an interest period

- Many bank accounts have an interest period of one year, or at least one month. If you deposit an amount *within* an interest period, the bank will often use simple interest not compound interest.
- At the end of the period the amount you deposited wil be multiplied by the factor $1 + rt$, where t is the remaining fraction of the period.
- Example: A the end of each quarter, beginning on March 31, 1999, a person deposits 100 dollars in an account on which interest is paid annually at the rate 10 percent per year. How much is there in an account on December 31, 2001?

Internal rate of return

- A different criterion to compare alternative investment project is based on the internal rate of return, defined as an interst rate that makes the present value of all payments equal to zero.
- For a general investment project yielding returns $a_0, a_1, ..., a_n$, the internal rate of return is a number r such that

$$a_0 + \frac{a_1}{1+r} + \frac{a_2}{(1+r)^2} + ... + \frac{a_n}{(1+r)^n} = 0$$

- Choose the project with the higher rate of return.
- Example: An investment project has an initial outlay of 50 000 TL, and at the end of the next two years has returns of 30 000 TL and 40 000 TL, respectively. Find the internal rate of return.