Izmir University of Economics
Econ 533: Quantitative Methods and Econometrics

One Variable Calculus

Introduction

- Finding the best way to do a specific task involves what is called an optimization problem.
- Studying an optimization problem requires mathematical methods of maximizing or minimizing a a function of a single variable.
- Useful economic insights can be gained from simple one variable optimization.

Introduction

- Those points in the domain of a function where it reaches its largest and smallest values are referred to as maximum and minimum points or **extreme points**. Thus, if $f(x)$ has domain D, then
  $c \in D$ is a maximum point for $f \Leftrightarrow f(x) \leq f(c)$ for all $x \in D$
  $d \in D$ is a minimum point for $f \Leftrightarrow f(x) \geq f(d)$ for all $x \in D$
- If the value of f is strictly larger than any other point in D, then c is a **strict maximum point**. Similarly, d is a **strict ninimum point** if $f(x) > f(d)$ for all $x \in D, x \neq d$.
- Example 1: Find possible maximum and minimum points for
  1. $f(x) = 3 - (x-2)^2$
  2. $g(x) = \sqrt{x-5} - 100, x \geq 5$

Necessary First Order Condition

Theorem

Suppose that a function f is differentiable in an interval I and that c is an interior point of I. For $x = c$ to be a maximum or a minimum point of F in I, a necessary condition is that it is a critical (stationary) point for f- i.e. that $x = c$ satisfies the equation

$$f'(x) = 0 \textbf{\textit{(first order condition)}}$$

First Derivative Test for Maximum/Minimum

Theorem

If $f'(x) \geq 0$ for $x \leq c$ and $f'(x) \leq 0$ for $x \geq c$, then $x = c$ is a maximum point for f.
If $f'(x) \leq 0$ for $x \leq c$ and $f'(x) \geq 0$ for $x \geq c$, then $x = c$ is a minimum point for f.

Example: Measured in miligrams per litre, the concentration of a drug in the bloodstream t hours after injection is given by the formula

$$c(t) = \frac{t}{t^2 + 4}, \quad t \geq 0$$

Find the time of maximum concentration.

Convex and concave functions

$f''(x) \geq 0$ on $I \Leftrightarrow f'$ is increasing on I
$f''(x) \leq 0$ on $I \Leftrightarrow f'$ is decreasing on I
Assuming that f is continuous in the interval I and twice differentiable in the interior of I:
f is convex on I $\Leftrightarrow f''(x) \geq 0$ for all x in I
f is concave on I $\Leftrightarrow f''(x) \leq 0$ for all x in I
Maximum/Minimum for Concave/Convex Functions

Theorem

Suppose that a function f is concave (convex) in an interval I. If c is a critical point of f in the interior of I, then c is a maximum (minimum) point of f in I.

Example: Show that $f(x) = e^{x-1} - 1$ is convex and find its minimum point.

Economic Examples 1

Example 1: Suppose $Y(N)$ bushels of wheat are harvested per acre of land when N pounds of fertilizer per acre are used. If P is the dollar price per bushel of wheat and q is the dollar price per pound of fertilizer, the profits in dollars per acre are

$$\pi(N) = PY(N) - qN, \quad N \geq 0$$

Suppose there exists N^* such that $\pi'(N) \geq 0$ for $N \leq N^*$, whereas $\pi'(N) \leq 0$ for $N \geq N^*$. Then N^* maximizes profits, and $\pi'(N^*) = 0$. That is, $PY'(N^*) - q = 0$, so

$$PY'(N^*) = q$$

In a constructed example $Y(N) = \sqrt{N}$, $P = 10$, and $q = 0.5$. Find the amount of fertilizer which maximizes profit in this case.

Example 2: The total cost of producing Q units of a commodity is

$$C(Q) = 2Q^2 + 10Q + 32, \quad Q > 0$$

Find the value of Q that minimizes the average cost.
The total cost of producing Q units of a commodity is

$$C(Q) = aQ^2 + bQ + c, \quad Q > 0$$

where a, b, and c are positive constants. Show that average cost function has a minimum at $Q^* = \sqrt{c/a}$.
Example 3: A monopolist is faced with the demand function $P(Q)$ denoting the price when output is Q. The monopolist has a constant average cost k per unit produced. Find the profit function $\pi(Q)$, and the first order condition for maximum profit.

The Extreme Value Theorem

Theorem

If f is a continuous function over a closed bounded interval $[a, b]$, then there exists a point d in $[a, b]$ where f has a minimum, and a point c in $[a, b]$ where f has a maximum, so that $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$

The recipe for finding maximum amd minimum values

Problem: Find the maximum and minimum values of a differentiable function f defined on a closed, bounded interval [a, b].

Solution:

1. Find all critical points of f in (a, b) - that is find all points x in (a, b) that satisfy equation $f'(x) = 0$.
2. Evaluate f at the end points a and b of the interval and at all stationary points.
3. The largest function value found in (2) is the maximum value and the smallest function value is the minimum value of f in $[a, b]$.

Example: Find the maximum and minimum values for

$$f(x) = 3x^2 - 6x + 5, \quad x \in [0, 3]$$

Further Economic Examples

Example: The total revenue of a single product firm is $R(Q)$ dollars, wheras $C(Q)$ is the associated dollar cost. Assume $\bar{Q}$ is the maximum quantity that can be produced and R and C are differentiable functions of Q in the interval $[0, \bar{Q}]$. The profit function is then differentiable, so continuous. Thus, π has a maximum value. In special cases, maximum might occur at $Q = 0$ or at $Q = \bar{Q}$. If not, it has an interior maximum where the production level Q^* satisfies $\pi'(Q^*) = 0$, and so

$$R^{'}(Q^*) = C^{'}(Q^*)$$

Production should be adjusted to a point where marginal revenue is equal to the marginal cost. The approximate extra revenue earned by selling extra unit is offset by the approximate extra cost of producing that unit.

Local Extreme Points

- Global optimization problems seek the largest or smallest values of a function at ALL points in the domain.
- Local optimization problems look at only the nearby points to find the the largest or smallest values of a function.
- The function f has a local maximum (minimum) at c, if there exists an interval (α, β) about c such that $f(x) \leq (\geq)\ f(c)$ for all x in (α, β) which are in the domain of f.
- *At a local extreme point in the interior of the domain of a differentiable function, the derivative must be zero.*

Local Extreme Points

- In order to find possible local maxima and minima for a function f defined in the interval I we search among the following types of point:
  i. Interior points in I where $f'(x) = 0$
  ii. End points of I
  iii. Interior points in I where f' does not exist
- How do we decide whether a point satisfying the necessary conditions is a local max, local min, or neither?

The First Derivative Test

Theorem

Suppose c is a stationary point for $y = f(x)$.

a. *If $f'(x) \geq 0$ throughout some interval (a, c) to the left of c and $f'(x) \leq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local maximum point for f.*

b. *If $f'(x) \leq 0$ throughout some interval (a, c) to the left of c and $f'(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point for f.*

c. *If $f'(x) > 0$ both throughout some interval (a, c) to the left of c and throughout some interval (c, b) to the right of c, then $x = c$ is not a local extremum point for f. The same condition holds if $f'(x) < 0$ on both sides of c.*

The Second Derivative Test

Theorem

Let f be a twice differentiable function in an interval I, and let c be an interior point of I. Then:

a. $f'(c) = 0$ *and* $f''(c) < 0 \Rightarrow x = c$ *is a strict local maximum point*

b. $f'(c) = 0$ *and* $f''(c) > 0 \Rightarrow x = c$ *is a strict local minimum point*

c. $f'(c) = 0$ *and* $f''(c) = 0 \Rightarrow$ *?*

Examples

1. Classify the stationary points of $f(x) = \frac{1}{9}x^3$ - $\frac{1}{6}x^2$-$\frac{2}{3}x + 1$ and $f(x) = x^2 e^x$ by using the first derivative test and second derivative test.
2. Suppose that the firm faces a sales tax of t dollars per unit. The firm's profit from producing and selling Q units is then

   $$\pi(Q) = R(Q) - C(Q) - tQ$$

   In order to maximize profits at some quantity Q^* satisfying $0 < Q^* < \bar{Q}$, one must have $\pi'(Q) = 0$. Hence,

   $$R'(Q) - C'(Q) - t = 0$$

   Suppose $R''(Q^*) < 0$ and $C''(Q^*) > 0$. This equation implicitly defines Q^* as a differentiable function of t. Find an expression for dQ^*/dt and discuss its sign.

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Inflection Points

- A **twice differentiable function** $f(x)$ is concave (convex) in an interval I with $f''(x) \leq 0(\geq)$ for all x in I.
- Points at which a function changes from being concave to convex, or vice versa, are called **inflection points**.
- For twice differentiable functions the definition is the following: The point c is called an inflection point for the function f if there exists an interval (a, b) about c such that:

  a. $f''(x) \geq 0$ in (a, c) and $f''(x) \leq 0$ in (c, b), or
  b. $f''(x) \leq 0$ in (a, c) and $f''(x) \geq 0$ in (c, b).

Test for Inflection Points

Theorem

Let f be a function with a continuous second derivative in an interval I, and let c be an interior point of I.

a. *If c is an inflection point for f, then $f''(c) = 0$.*

b. *If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.*

Examples

1. Show that $f(x) = x^4$ does not have an inflection point at $x = 0$, even though $f''(0) = 0$.
2. Find possible inflection points for $f(x) = x^6 - 10x^4$.

Production function

Suppose that $x = f(v)$, $v \geq 0$ is a production function. It is assumed that the function is S-shaped. That is the marginal product $f'(v)$ is increasing up to a certain production level v_0, and then decreasing. If f is twice differentiable, the $f''(v)$ is ≥ 0 in $[0, v_0]$ and ≤ 0 in $[v_0, \infty]$. Hence, f is first convex and then concave, with v_0 as an inflection point. Note that at v_0 a unit increase in input gives the maximum increase in output.

Strictly Concave and Convex Functions

- A function is strictly concave (convex) if the line segment joining any two points on the graph is strictly below (above) the graph
- Obvious sufficient conditions for strict concavity/convexity are the following:
  - $f''(x) < 0$ for all $x \in (a, b) \Rightarrow f(x)$ is strictly concave in (a, b)
  - $f''(x) > 0$ for all $x \in (a, b) \Rightarrow f(x)$ is strictly convex in (a, b)
- The reverse implications are not correct. For instance, $f(x) = x^4$ is strictly convex in the interval $(-\infty, \infty)$, but $f''(x)$ is not > 0 everywhere because $f''(0) = 0$.

Implicit Differentiation

- Consider the following equation: $xy = 5$.
- In general, for each number $x \neq 0$, there is a unique number y such that the pair (x, y) satisfies the equation. We say that this equation defines y implicitly as a function of x.
- Economists need to know the slope of the tangent at an arbitrary point on such a graph.
- The answer can be found by implicit differentiation of the equation, which defines y as a function of x.

The Method of Implicit Differentiation

If two variables x and y are related by an equation, to find y':

a. Differentiate each side of the equation w.r.t. x, considering y as a function of x.

b. Solve the resulting equation for y'.

Economic Examples

- Example 1: In a standard macroeconomic model for determining national income in a closed economy, its is assumed that

$$Y = C + I \quad (1)$$
$$C = f(Y) \quad (2)$$

Assume that $f'(Y)$, marginal propensity to consume, is between 0 and 1.

a. Suppose first that $C = f(Y) = 95.05 + 0.712Y$, and use the equations above to find Y in terms of I.

b. Inserting the expression for C from (2) to (1) gives $Y = f(Y) + I$. Suppose that this function defines Y as a differentiable function of I. Find an expression for dy/dI.

c. Find $'' = d^2Y/dI^2$

Economic Examples

- Example 2: In the linear supply and demand model, a tax is imposed on consumers. Then,

$$\begin{aligned} D &= a - b(P + T) \\ S &= \alpha + \beta P \end{aligned}$$

Here a, b, α, and β are positive constants. The equilibrium price is determined by equating supply and demand, so that

$$a - b(P + T) = \alpha + \beta P \tag{3}$$

Equation above implicitly defines the price P as a function of the unit tax t. Compute dP/dT by implicit differentiation. What is its sign? What is the sign of $\frac{d}{dT}(P + T)$? Check the result by first solving equation (3) and then finding dP/dT explicitly.

Elasticities

- Economists want a measure of sensitivity of the demand to price changes which can not be manipulated by choice of units.
- The solution to this problem is to use percent change instead of the actual change. For any quantity, the percent rate of change is the actual change divided by the initial amount:

$$\frac{q_1 - q_0}{q_0} = \frac{\Delta q}{q_0}.$$

- Since the numerator and the denominator are measured in the same units, the units cancel out in the division process.
- Price elasticity of demand is the percent change in demand for each 1 percent rise in price.

$$\epsilon = \frac{\Delta x}{x} / \frac{\Delta p}{p} = \frac{\Delta x}{x} . \frac{p}{\Delta p} = \frac{\Delta x}{\Delta p} . px = \frac{\Delta x}{\Delta p} / \frac{x}{p} \tag{4}$$

Elasticities

- Elasticity is marginal demand divided by average demand. Substituting $F'(p)$ for $\Delta x/\Delta p$ and $F(p)$ for x yields the calculus form of the price elasticity:

$$\epsilon = \frac{F'(p)p}{F(p)}. \tag{5}$$

- The discrete version (4) of the price elasticity is called the **arc elasticity** and is used when we know only a number of combinations.
- The differentiable (5) of the price elasticity is called the **point elasticity** and is used when a continuous demand curve is estimated.

Elasticities

Theorem

For an inelastic (elastic) good, an increase in price leads to an increase (decrease) in total expenditure.

Proof?

- Economists use specific functional forms for the demand functions, especially linear demand

$$x = F(p) \equiv a - bp, \quad a, b > 0$$

- and constant elasticity demand

$$x = F(p) \equiv kp^{-r}, \quad k, r > 0$$

Elasticities

- Since the slope of F differs from the elasticity of F, the elasticity varies along a demand curve:

$$\epsilon = \frac{F'(p)p}{F(p)} = \frac{-bp}{a - bp} = \frac{1}{1 - (a/bp)}$$

- from $\epsilon = 0$ when $p = 0$ and $x = a$ to $\epsilon = -\infty$ when $p = a/b$ and $x = 0$.
- Example: Assume that the quantity demanded of a particular commodity is given by

$$D(p) = 8000p^{-1.5}$$

  Compute the elasticity of $D(p)$ and find the exact percentage change in quantity demanded when the price increases by 1 percent from $p = 4$.

Intermediate Value Theorem: Newton Method

Theorem

Let f be a function of continuous in $[a, b]$ and assume that $f(a)$ and $f(b)$ have different signs. Then there is at least one c in (a, b) such that $f(c) = 0$.

Newton's Method

- Consider the graph of $y = f(x)$, which has a zero at $x = a$, but this zero is not known. To find it, start with initial estimate, x_0, of a. In order to improve the estimate, construct the tangent line to the graph at the point $(x_0, f(x_0))$, and find the point x_2 at which the tangent crosses the x-axis.
- x_1 is a significantly better estimate of a than x_0. Repeat the procedure by constructing the tangent line to the graph at the point $(x_1, f(x_1))$. Find the point x_1 at which the tangent crosses the x-axis. Repeating this procedure leads to a sequence of points which converges to a.

Newton's Method

- The equation for the tangent-line through the point $(x_0, f(x_0)$ with slope $f'(x_0)$ is given by
$$y - f(x_0) = f'(x_0)(x - x_0)$$
- At the point where this tangent line crosses the x-axis, $y = 0$ and $x = x_1$. Thus $-f(x_0) = f'(x_0)(x_1 - x_0)$. Solving this equation for x_1, we get
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
- In general, one has the following formula
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots$$
- Usuallly, the sequence x_n converges quickly to a zero of f.

Examples

- Example 1: Find an approximate value for the zero of $f(x) = x^6 + 3x^2 - 2x - 1$, in the interval $[0, 1]$, using Newton's method once.
- Example 2: Use Newton's method twice to find an approximate value for $\sqrt[15]{2}$.