

Profit Maximizing Firm: Suppose that $\theta = f(K, L)$ is a production function with K as the capital input and L as the labor input. The price per unit of output is p , the (or rental) per unit of capital is r , and the wage rate is w . The constants p, r , and w are all positive. The profit Π from producing and selling $f(K, L)$ units is then given by the function

$$\Pi(K, L) = p f(K, L) - rK - wL$$

~~If f is differentiable and~~

Assume that the profit maximizing firm uses a positive amount of each input and f is differentiable. Then, the first order conditions are

$$\frac{\partial \Pi(K, L)}{\partial K} = p \frac{\partial f(K, L)}{\partial K} - r = 0$$

$$\frac{\partial \Pi(K, L)}{\partial L} = p \frac{\partial f(K, L)}{\partial L} - w = 0$$

Thus, a necessary condition for profit to be a maximum when $K=K^*$ and $L=L^*$ is that

$$pf'_K(K^*, L^*) = r, \quad pf'_L(K^*, L^*) = w.$$

If we increase capital input from level K^* by 1 unit, we would increase production by approximately $f'_K(K^*, L^*)$ units. Because each of these units is priced at p , the gain in revenue is approximately $p f'_K(K^*, L^*)$. On the other hand, we lose r by increasing capital input by one unit. Recall that r is the price of ^{one unit of} capital. Thus, the profit maximizing K^* ~~not~~ has the property that extra revenue from increasing capital by one unit is just

offset by the extra cost of capital. The second equation for labor has a similar interpretation.

We can formulate the FOCs in the alternative form:

$$\underbrace{f'_K(K^*, L) = \frac{r}{P}}_{\text{marginal product of capital}} \quad \underbrace{f'_L(K^*, L) = \frac{w}{P}}_{\text{the relative price of capital}}$$

$$\underbrace{\text{marginal product of capital}}_{f'_K(K^*, L)} \quad \underbrace{\text{the relative price of capital}}_{\frac{w}{P}}$$

So, to obtain maximum profit, the firm must choose K and L such that the marginal product of Capital is equal to the relative price of Capital, and marginal product of Capital labor is equal to the relative price of labor.

Note that these conditions are necessary, but generally not sufficient for an interior maximum.

Now suppose that F is twice differentiable. Sufficient conditions for an optimum :

$$D^2F(K^*, L) = \begin{pmatrix} \frac{\partial^2 F}{\partial K^2} & \frac{\partial^2 F}{\partial K \partial L} \\ \frac{\partial^2 F}{\partial K \partial L} & \frac{\partial^2 F}{\partial L^2} \end{pmatrix}$$

$K > 0$ and $L >$

If $D^2F(K^*, L)$ is negative semidefinite for all ~~except zero~~

f is concave. Then (K^*, L^*) maximizes profit.

The first leading principal minor (LPM) $\frac{\partial^2 F}{\partial K^2} \leq 0$

The second LPM $\frac{\partial^2 F}{\partial K^2} \frac{\partial^2 F}{\partial L^2} - \left(\frac{\partial^2 F}{\partial K \partial L} \right)^2 > 0$

for ALL $K > 0$ and $L > 0$.

Discriminating Monopolist: Consider a firm that sells a product in two separate markets, for example, a domestic market and a foreign market, each with its own demand function. Let Q_1 be the amount supplied to market 1 and Q_2 be the amount supplied to market 2. The inverse demand functions for the two markets are $50 - 5Q_1$ and $100 - 10Q_2$ dollar per unit. The manufacturing cost is $90 + 20(Q_1 + Q_2)$. Its profit is then

$$P(Q_1, Q_2) = Q_1(50 - 5Q_1) + Q_2(100 - 10Q_2) - (90 + 20 \cdot (Q_1 + Q_2))$$

And the critical points satisfying

$$\frac{\partial P}{\partial Q_1} = 50 - 10Q_1 - 20 = 0 \Leftrightarrow Q_1 = 3$$

$$\frac{\partial P}{\partial Q_2} = 100 - 20Q_2 - 20 = 0 \Leftrightarrow Q_2 = 4$$

We check 2nd order conditions

$$D^2P = \begin{bmatrix} \frac{\partial^2 P}{\partial Q_1^2} & \frac{\partial^2 P}{\partial Q_1 \partial Q_2} \\ \frac{\partial^2 P}{\partial Q_1 \partial Q_2} & \frac{\partial^2 P}{\partial Q_2^2} \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & -20 \end{bmatrix}$$

The first order leading principal minor (LPM) of $D^2F(3,4)$ is $-10 < 0$ and the second order LPM is 200 .

Please notice that the Hessian is negative definite for EVERY simple Q_1 and Q_2 not only for $(3,4)$. Therefore, P is concave and hence $(Q_1^*, Q_2^*) = (3,4)$ is a global max.