Izmir University of Economics
Econ 533: Quantitative Methods and Econometrics

Constrained Optimization II

Introduction

- We look at two aspects of the Lagrangian approach
  1. the sensitivity of the optimal value of the objective function to changes in the paremetrs of the problem
  2. the second order conditions that distinguish maxima from minima

The Meaning of the Multiplier

One Equality Constraint

$$\textit{maximize} \qquad f(x, y)$$

$$\textit{subject to} \qquad h(x, y) = a$$

- a is a parameter. For any a, write $(x^*(a), y^*(a))$ for the solution to the problem and write $\mu^*(a)$ for the multiplier. Let $f(x^*(a), y^*(a))$ be the optimal value of the objective function.
- Under reasonable conditions, $\mu^*(a)$ measures the rate of change of the optimal value of f with respect to the parameter a, **the (infinitesimal) effect of a unit increase in** a on $f(x^*(a), y^*(a))$.

Theorem

Let f and h be C^1 functions of two variables. For any fixed value of the parameter a, let $(x^*(a), y^*(a))$ be the solution of problem with corresponding multiplier $\mu^*(a)$. Suppose that (x^*, y^*, and μ^*) are C^1 functions and that NDCQ holds at $(x^*(a), y^*(a), \mu^*(a))$. Then,

$$\mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a)).$$

Several Equality Constraints

Theorem

Let $f, h_1, \ldots, h_m$ *be* C^1 *functions on* $\mathbf{R^n}$*. Let* $a = (a_1, \ldots, a_m)$ *be an m-tuple of exogenous parameters, and consider the problem* $P(_a)$ *of maximizing* $f(x_1, \ldots, x_n)$ *subject to the constraints*

$$h_1(x_1, \ldots, x_n) = a_1, \ldots, h_m(x_1, \ldots, x_n) = a_m.$$

Let $x_1^*(\mathbf{a}), \ldots, x_n^*(\mathbf{a})$ *denote the solution of the problem* (P_a)*, with corresponding Lagrange multipliers* $\mu_1^*(\boldsymbol{a}), \ldots, \mu_m^*(\boldsymbol{a})$*. Suppose futher that the* x_i^**'s and* μ_j^**'s are differentiable functions of* $(a_1, \ldots, a_m)$ *and that NDCQ holds. Then for each* $j = 1, \ldots, m$*,*

$$\mu_j^*(a_1, \ldots, a_m) = \frac{\partial}{\partial a_j} f(x_1^*(a_1, \ldots, a_m), \ldots, x_n^*(a_1, \ldots, a_m)).$$

Inequality Constraints

Theorem

Let $\mathbf{a}^* = (a_1^*, \ldots, a_k^*)$ be k-tuple. Consider the problem (Q_{a^*}) of maximizing $f(x_1, \ldots, x_n)$ subject to the k inequality constraints

$$g_1(x_1, \ldots, x_n) \leq a_1^*, \ldots, g_k(x_1, \ldots, x_n) \leq a_k^*.$$

Let $x_1^*(\mathbf{a}^*), \ldots, x_n^*(\mathbf{a}^*)$ denote the solution of the problem (Q_{a^*}), with corresponding Lagrange multipliers $\lambda_1^*(\mathbf{a}^*), \ldots, \lambda_k^*(\mathbf{a}^*)$. Suppose futher that as $\mathbf{a}$ varies near $\mathbf{a}^*, x_1^*, \ldots, x_n^*$ and $\lambda_1^*, \ldots, \lambda_k^*$ are differentiable functions of $(a_1, \ldots, a_k)$ and that the NDCQ holds at $\mathbf{a}^*$. Then, for each $j = 1, \ldots, k$,

$$\lambda_j^*(a_1^*, \ldots, a_k^*) = \frac{\partial}{\partial a_j} f(x_1^*(a_1^*, \ldots, a_k^*), \ldots, x_n^*(a_1^*, \ldots, a_k^*)).$$

Interpreting the Multiplier

- The multiplier measures the sensitivity of the optimal value of the objective function to the changes in **a**, the constraint constant (the right hand side of the constraint).
- In economic applications, **a** denotes the available stock of some resource, and the objective function denotes utility or profit. Then $\mu(a)da$ measures the approximate change in utility or profit that can be obtained from da units more (or $-da$, when $da < 0$). If $f^*(a)$ is the maximum profit when the resource input is a, choosing $da = 1$ gives the approximation $f^*(a+1) - f^*(a) \approx \mu(a)$. This means that μ indicates approximately by how much profits increase if one more unit of the resource is made avalable.

Envelope Theorems

Theorems which describe how the optimal value of the objective function in a **parameterized** optimization problem changes as one of the parameters changes.

Unconstrained Problems

Theorem

Let $f(\mathbf{x}; a)$ be a C^1 function of $\mathbf{x} \in R^n$ and the scalar a. For each choice of the parameter a, consider the unconstrained maximization problem

$$\textit{maximize} \qquad f(\mathbf{x}; a) \qquad \textit{with respect to} \quad \mathbf{x}$$

Let $\mathbf{x}^*(a)$ be a solution of this problem. Suppose that $\mathbf{x}^*(a)$ is a C^1 function of a. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} f(\mathbf{x}^*(a); a)$$

- It is a very useful result because partial derivative of the right-hand side is easier to deal with than the total derivative on the left-hand side.
- When a changes, then f^* changes for two reasons:
    1. A change in a changes $f(x, a)$ directly
    2. A change in a changes $x^*(a)$, and so $f(x^*(a), a)$ changes indirectly.
- The result in the previous theorem shows that the total effect on the optimal value of the objective function of a small change in a is found by computing the partial derivative of $f(x, a)$ with respect to a, and evaluating it at $x^*(a)$, ignoring the indirect effect of the dependence of x^* on a altogether. The reason is that any small change in $\mathbf{x}$ has a negligible effect on the value of $f(x^*, a)$.

Constrained Problems

Theorem

Let $f, h_1, \ldots, h_k : \mathbf{R^n} \text{ x } \mathbf{R^1} \rightarrow \mathbf{R^1}$ be C^1 functions. Let $\mathbf{x}^*(\mathbf{a}) = x_1^*(a), \ldots, x_n^*(a)$ denote the solution of the problem of maximizing $\mathbf{x} \mapsto f(\mathbf{x}; a)$ on the constraint set

$$h_1(\mathbf{x}; a) = 0; \ldots, h_k(\mathbf{x}; a) = 0,$$

for any fixed choice of the parameter a. Suppose that $x(a)$ and the Lagrange multipliers $\mu_1(a), \ldots, \mu_k(a)$ are C^1 functions of a and that NDCQ holds. Then,

$$\frac{d}{da} f(\mathbf{x}^*(a); a) = \frac{\partial}{\partial a} L(\mathbf{x}^*(a); \mu(a); a),$$

where L is the natural Lagrangian for this problem.

Second Order Conditions

- Second order conditions help us choose a maximizer from the set of candidates which satisfy the first order conditions.
- The SOC for maximizing an unconstrained function $f(x_1, ..., x_n)$ is that the Hessian of f at the maximizer x^*

$$\mathbf{D^2f(x^*)} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}^*) & \dots & \frac{\partial f^2}{\partial x_n x_1}(\mathbf{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^2}{\partial x_1 x_n}(\mathbf{x}^*) & \dots & \frac{\partial f^2}{\partial x_n^2}(\mathbf{x}^*) \end{pmatrix}$$

  be negative definite.
- At a maximum $f(x^*)$, $Df(x^*)$ must be zero and $D^2f(x^*)$ must be negative semidefinite (necessary conditions).
- To guarantee that a point x^* is a local maximizer, we need $Df(x^*) = 0$ and $D^2f(x^*)$ negative (sufficient conditions).

- Remember the condition on bordered matrices for verifying SOC. Border the *nxn* Hessian $D^2L(x^*, \mu^x)$ by the *kxn* constraint matrix $Dh(x^*)$:
- $$\mathbf{H} \equiv \begin{pmatrix} 0 & D\mathbf{h}(\mathbf{x}^*) \\ D\mathbf{h}(\mathbf{x}^*)^T & D^2L(\mathbf{x}^*, \mu^*) \end{pmatrix}$$

$$\mathbf{D^2L} = \begin{pmatrix} 0 & \dots & 0 & \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial h_k}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_k}{\partial x_1} & \frac{\partial^2 L}{\partial x_1^2} & \dots & \frac{\partial^2 L}{\partial x_n x_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \dots & \frac{\partial h_k}{\partial x_n} & \frac{\partial^2 L}{\partial x_1 x_n} & \dots & \frac{\partial^2 L}{\partial x_n^2} \end{pmatrix}$$

If det H has the same sign as $(-1)^n$ and if these last $(n-k)$ leading principal minors of matrix alternate in sign with the sign of det H, H is negative definite.

Theorem

Let $f, h_1, \ldots, h_k$ be C^2 functions on $\mathbf{R^n}$. Consider the problem of maximizing f on the constraint set

$$C_h \equiv \{\mathbf{x} : h_1(\mathbf{x}) = c_1, \ldots, h_k(\mathbf{x}) = c_k\}.$$

Form the Lagrangian, and suppose that

1. $\mathbf{x}^*$ *lies in the constraint set* C_h,
2. *there exist* $\mu_1^*, \ldots, \mu_k^*$ *such that*

   $$\frac{\partial L}{\partial x_1} = 0, \ldots, \frac{\partial L}{\partial x_n} = 0, \frac{\partial L}{\partial \mu_1} = 0, \ldots, \frac{\partial L}{\partial \mu_k} = 0$$

   at $(x_1^*, \ldots, x_n^*, \mu_1^*, \ldots, \mu_k^*)$.
3. *the Hesssian of L with respect to* $\mathbf{x}$ *at* $(\mathbf{x}^*, \mu^*)$, $D^2 L(\mathbf{x}^*, \mu^*)$ *is negative definite. Then,* x^* *is a strict local constrained max*

Simplest max. problem: two variables and one equality constraint

Theorem

Let f and h be C^2 functions on $\mathbf{R^2}$. Consider the problem of maximizing f on the constraint set $C_h = (x, y) : h(x, y) = c$. Form the Lagrangian

$$L(x, y, \mu) = f(x, y) - \mu(h(x, y) - c).$$

Suppose that (x^*, y^*, μ^*) satisfies:

$$\frac{\partial L}{\partial x} = 0, \frac{\partial L}{\partial y} = 0, \frac{\partial L}{\partial \mu} = 0$$

at (x^*, y^*, μ^*), and

$$det \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} \end{pmatrix}$$

is positive at (x^*, y^*, μ^*).

Then, (x^*, y^*) is a local *max* of f on C_h.